Math 591 – Homework 2
Due 1:30pm on Thursday, February 11, 2016

Please indicate any sources you used to find the solution to a given problem. For example, if you worked with another student to get the solution to a problem, please indicate who. You are welcome to work together in small groups, but try the problems on your own first and write up your own solutions.

Problem 1. Let $f, g \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be Laurent polynomials.

(a) Prove that for any $w \in \mathbb{R}^n$, $\text{in}_w(f \cdot g) = \text{in}_w(f) \cdot \text{in}_w(g)$.

(b) Show that the Newton polytope of $f \cdot g$ is the Minkowski sum of their Newton polytopes, i.e.

$$\text{Newt}(f \cdot g) = \text{Newt}(f) + \text{Newt}(g),$$

where the Minkowski sum of two polytopes $A, B$ is $A + B = \{a + b : a \in A, b \in B\}$.

(c) How does $\mathcal{T}(\text{trop}(f \cdot g))$ relate to $\mathcal{T}(\text{trop}(f))$ and $\mathcal{T}(\text{trop}(g))$?

Problem 2. Let $f(x, y) \in \mathbb{Q}[x_1, \ldots, x_n, y]$ and $F = \text{trop}(f)$, taking $\mathbb{Q}$ with the trivial valuation. Define $g = f(x, t) \in \mathbb{Q}(t)[x_1, \ldots, x_n]$ and $G = \text{trop}(g)$. Show that

$$\mathcal{T}(G) = \{ w \in \mathbb{R}^n \text{ such that } (w, 1) \in \mathcal{T}(F) \}.$$  

Problem 3. Let $K$ be an algebraically closed field and take an ideal $I \subset K[x_1, \ldots, x_n]$.
Show that $V(I) \cap (K^*)^n$ is empty if and only if $I$ contains a monomial.
(Hint: Use Hilbert’s Nullstellensatz)

Problem 4. Consider the ideal $I = \langle f, g \rangle \subset \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$ where

$$f = t^2 x^2 + xy + t^2 y^2 + x + y + t^2 \quad \text{and} \quad g = 5 + 6tx + 17ty - 4t^3 xy.$$ 
Let $F = \text{trop}(f)$ and $G = \text{trop}(g)$.

(a) For each $w \in \mathcal{T}(F) \cap \mathcal{T}(G)$, compute $\text{in}_w(I)$.

(b) Is $\{f, g\}$ a tropical basis for $I$?

(c) There are four points in the variety $V(\langle f, g \rangle) \subset (\mathbb{C}\{\{t\}\}^*)^2$.
Compute the leading term of each.