Eulerian polynomials, chromatic quasisymmetric functions, and Hessenberg varieties

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Joint work with John Shareshian

Eulerian Polynomials

For $\sigma \in \mathfrak{S}_n$

$$\begin{aligned} \mathrm{DES}(\sigma) &:= \{i \in \{1, \ldots, n-1\} : \sigma(i) > \sigma(i+1)\} \\ \mathrm{des}(\sigma) &:= |\mathrm{DES}(\sigma)| \end{aligned}$$

For $\sigma = 3.25.4.1$

$$\mathrm{DES}(\sigma) = \{1, 3, 4\} \qquad \mathrm{des}(\sigma) = 3$$

Eulerian polynomial

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)}$$

$A_1(t) = 1$
$A_2(t) = 1 + t$
$A_3(t) = 1 + 4t + t^2$
$A_4(t) = 1 + 11t + 11t^2 + t^3$
$A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$

- *palindromic*
- *unimodal*

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Eulerian Polynomials

A new formula?

$$A_n(t) = \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{k_1,\ldots,k_m \geq 2} \binom{n}{k_1 - 1, k_2, \ldots, k_m} t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t$$

where

$$[k]_t := 1 + t + \cdots + t^{k-1}.$$

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Sum & Product Lemma: Let $A(t)$ and $B(t)$ be positive, unimodal, palindromic with respective centers of symmetry c_A and c_B. Then

- $A(t)B(t)$ is positive, unimodal, and palindromic with center of symmetry $c_A + c_B$.
- If $c_A = c_B$ then $A(t) + B(t)$ is positive, unimodal and palindromic with center of symmetry c_A.

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Center of symmetry:

$$(m-1) + \sum_{i=1}^{m} \frac{k_i - 2}{2} = \frac{1}{2}(n-1).$$

$$
\begin{aligned}
1+\sum_{n\geq 1} A_n(t)\frac{z^n}{n!} &= \frac{1-t}{e^{z(t-1)}-t} \qquad \text{(Euler's exponential generating function formula)}\\
&= \frac{(1-t)\exp(z)}{\exp(zt)-t\exp(z)}\\
&= \exp(z)\frac{1-t}{\sum_{k\geq 0}\frac{t^k z^k - tz^k}{k!}}\\
&= \exp(z)\frac{1-t}{\sum_{k\geq 0}\frac{(t^k-t)z^k}{k!}}\\
&= \left(\sum_{r\geq 0}\frac{z^r}{r!}\right)\left(1-\sum_{k\geq 2}\frac{t[k-1]_t z^k}{k!}\right)^{-1}\\
&= \left(\sum_{r\geq 0}\frac{z^r}{r!}\right)\sum_{m\geq 0}\left(\sum_{k\geq 2}\frac{t[k-1]_t z^k}{k!}\right)^{m}\\
A_n(t) &= \sum_{m=0}^{\lfloor \frac{n}{2}\rfloor}\sum_{r=0}^{n}\sum_{k_1,\ldots,k_m\geq 2}\binom{n}{r,k_1,\ldots,k_m} t^m\prod_{i=1}^{m}[k_i-1]_t\\
&= \text{further manipulations}\\
&= \sum_{m=1}^{\lfloor \frac{n+1}{2}\rfloor}\sum_{k_1,\ldots,k_m\geq 2}\binom{n}{k_1-1,k_2,\ldots,k_m} t^{m-1}\prod_{i=1}^{m}[k_i-1]_t
\end{aligned}
$$

Mahonian Permutation Statistics - q-analog

Let $\sigma \in \mathfrak{S}_n$.

Inversion Number:

$$\mathrm{inv}(\sigma) := |\{(i,j) : 1 \leq i < j \leq n, \quad \sigma(i) > \sigma(j)\}|.$$

$$\mathrm{inv}(32541) = 6$$

Major Index:

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{DES}(\sigma)} i$$

$$\mathrm{maj}(32541) = \mathrm{maj}(3.25.4.1) = 1 + 3 + 4 = 8$$

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Theorem (MacMahon 1905)

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q!$$

where $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$

Rawlings major index

Let $1 \leq r \leq n$. For $\sigma \in \mathfrak{S}_n$, set

$$\begin{aligned}
\mathrm{inv}_{<r}(\sigma) &:= |\{(i,j) : 1 \leq i < j \leq n, \quad 0 < \sigma(i) - \sigma(j) < r\}|. \\
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\mathrm{maj}_{\geq r}(\sigma) &:= \sum_{i \in \mathrm{DES}_{\geq r}} i
\end{aligned}$$

Note

$$\mathrm{maj}_{\geq r} + \mathrm{inv}_{<r} = \begin{cases} \mathrm{maj} & \text{if } r = 1 \\ \mathrm{inv} & \text{if } r = n \end{cases}$$

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Theorem (Rawlings, 1981)

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$\mathrm{inv}_{<2}(635142) = 3$ since (<2)-inversions are

$$6\mathbf{3}514\mathbf{2} \qquad 63\mathbf{5}1\mathbf{4}2 \qquad \mathbf{6}3\mathbf{5}142$$

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$\mathrm{inv}_{<2}(\sigma) = \mathrm{des}(\sigma^{-1})$ $\qquad$ $A_n^{(2)}(1,t) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = A_n(t)$

So $A_n^{(r)}(1,t)$ is a generalized Eulerian polynomial and $A_n^{(r)}(q,qt)$ is a Mahonian q-analog.

Generalized Eulerian polynomial $A_n^{(r)}(t) := A_n^{(r)}(1, t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}_{<r}(\sigma)}$

Eulerian polynomials are palindromic and unimodal.

$$A_3^{(2)}(t) = 1 + 4t + t^2$$

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$$A_n^{(n-1)}(t) = [n-2]_t!\left([n]_t[n-2]_t + nt^{n-2}\right)$$

center of symmetry: $\frac{1}{2}((n-1) + (n-3)) = n-2$

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Problem (Stanley EC1, 1.50 f): Prove that $A_n^{(r)}(t)$ is palindromic and unimodal.

Solution:

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Solution:

Theorem (De Mari and Shayman (1988))

$A_n^{(r)}(t)$ is palindromic and unimodal for all $r \in [n]$.

- Palindromicity: easy
- Unimodality: They show that $A_n^{(r)}(t)$ is the Poincaré polynomial of a Hessenberg variety and apply hard Lefschetz theorem.

Stanley: Is there a more elementary proof?

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Is there a q-analog of this result?

q-analog: $A_n^{(r)}(q,t) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}_{\geq r}(\sigma)} t^{\mathrm{inv}_{<r}(\sigma)}$

A polynomial $\sum_{j=0}^{n} f_j(q) t^j \in \mathbb{Z}[q][t]$ is q-unimodal if

$$f_0(q) \leq_q f_1(q) \leq_q \cdots \leq_q f_c(q) \geq_q \cdots \geq_q f_{n-1}(q) \geq_q f_n(q),$$

where $f(q) \leq_q g(q)$ means $g(q) - f(q) \in \mathbb{N}[q]$.

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Proposition

$A_n^{(r)}(q,t)$ is palindromic as a polynomial in t for all $r \in [n]$.

Conjecture (Shareshian and MW)

$A_n^{(r)}(q,t)$ is q-unimodal for all $r \in [n]$.

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$$A_n^{(n-1)}(q,t) = [n-2]_t\left([n]_t[n-2]_t + [n]_q\, t^{n-2}\right)$$

q-Eulerian polynomials, $r = 2$:

$A_n^{(2)}(q,t) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}_{\geq 2}(\sigma)} t^{\mathrm{inv}_{<2}(\sigma)}$

$n \backslash j$	0	1	2	3	4
1	1				
2	1	1			
3	1	$2+q+q^2$	1		
4	1	$3+2q+3q^2+2q^3+q^4$	$3+2q+3q^2+2q^3+q^4$	1	
5	1	$4+3q+5q^2+\ldots$	$6+6q+11q^2+\ldots$	$4+3q+5q^2+\ldots$	1

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Theorem (Shareshian and MW)

$$A_n^{(2)}(q,t) = \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{k_1,\dots,k_m \geq 2} \begin{bmatrix} n \\ k_1-1, k_2, \dots, k_m \end{bmatrix}_q t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t$$

To prove this we use theory of P-partitions and quasisymmetric functions to get a q-analog of Euler's exponential generating function formula.

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Shareshian & MW: $A_n^{(2)}(q,t) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$

Formulae

r	$A_n^{(r)}(q,t)$
1	$[n]_q!$
2	$\sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{k_1,\ldots,k_m \geq 2} \begin{bmatrix} n \\ k_1-1, k_2, \ldots, k_m \end{bmatrix}_q t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t$
$\vdots$	???
$n-2$	$[n]_t[n-3]_t![n-3]_t^2 + [n]_q\ t^{n-3}[n-4]_t![n-2]_t([n-3]_t + [2]_t[n-4]_t) + \begin{bmatrix} n \\ n-2, 2 \end{bmatrix}_q t^{3n-10}[n-4]![n-2]_t[2]_t$
$n-1$	$[n]_t[n-2]_t![n-2]_t + [n]_q t^{n-2}[n-2]_t!$
n	$[n]_t!$

The q-unimodality conjecture - again

Conjecture (Shareshian and MW)

$A_n^{(r)}(q,t)$ is q-unimodal for all $r \in [n]$.

True for $q = 1$ Hessenberg varieties

True for $1 \leq r \leq 2$ & $n - 2 \leq r \leq n$ Sum & Product Lemma

Hessenberg Varieties (De Mari and Shayman - 1988)

Let $\mathcal{F}$ be the set of all flags

$$F : V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$$

where $\dim V_i = i$. Fix $M \in GL_n(\mathbb{C})$ with n distinct eigenvalues.

The type A regular semisimple Hessenberg variety of degree r is

$$\mathcal{H}_{n,r} := \{F \in \mathcal{F} \mid MV_i \subseteq V_{i+r-1} \text{ for all } i\}$$

Theorem (De Mari and Shayman - 1988)

$$A_n^{(r)}(1,t) = \sum_{j=0}^{d(n,r)} \dim H^{2j}(\mathcal{H}_{n,r}) t^j$$

Consequently by the hard Lefschetz theorem, $A_n^{(r)}(1,t)$ *is palindromic and unimodal.*

The hard Lefschetz theorem

Theorem (Hard Lefschetz Theorem)

Let Y be a smooth irreducible complex projective variety of (complex) dimension m. Then for some $\omega \in H^2(Y)$ and all $j = 0, \ldots, m$, the map $H^j(Y) \to H^{2m-j}(Y)$, given by multiplication by ω^{m-j} in the singular cohomology ring $H^*(Y)$, is a vector space isomorphism.

It follows that the map $H^j(Y) \to H^{j+2}(Y)$ given by multiplication by ω is injective. Hence for all $j = 0, \ldots, m$,

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$$A_n^{(r)}(1, t) = \sum_{j=0}^{d(n,r)} \dim H^{2j}(\mathcal{H}_{n,r}) t^j$$

From representations to q-analogs

$$\{\text{Representations of } \mathfrak{S}_n\} \xrightarrow{\text{ch}} \Lambda^n_{\mathbb{Z}} \xrightarrow{\text{ps}} \mathbb{Z}[q]$$

$\Lambda^n_{\mathbb{Z}}$: homogeneous symmetric functions over $\mathbb{Z}$ of degree n

ch: Frobenius characteristic

ps: stable principal specialization.

$$ps(f(x_1, x_2, \dots)) = f(1, q, q^2, \dots) \prod_{i=1}^{n} (1 - q^i)$$

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Tymoczko (2008) used a theory of Goresky, Kottwitz and MacPherson (GKM theory) to define a representation of $\mathfrak{S}_n$ on $H^{2j}(\mathcal{H}_{n,r})$.

Tymoczko's representation

MacPherson & Tymoczko show that the hard Lefschetz map commutes with the action of $\mathfrak{S}_n$ on $H^*(\mathcal{H}_{n,r})$. This means that the hard Lefschetz map $H^j(\mathcal{H}_{n,r}) \to H^{j+2}(\mathcal{H}_{n,r})$ is an $\mathfrak{S}_n$-module injection for all $j = 0, \ldots, d(n, r)$.

$\Rightarrow$ $\mathrm{ch} H^{j+2}(\mathcal{H}_{n,r}) - \mathrm{ch} H^j(\mathcal{H}_{n,r})$ is Schur-positive.

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Conjecture (Shareshian and MW)

$$A_n^{(r)}(q,t) = \sum_{j=0}^{d(n,r)} \mathrm{ps}(\mathrm{ch} H^{2j}(\mathcal{H}_{n,r})) t^j$$

Chromatic quasisymmetric functions

Let G be a graph on $[n]$. The chromatic quasisymmetic function $X_G(\mathbf{x}, t)$ is a polynomial in t whose coefficients are quasisymmetric functions. It is a refinement of Stanley's chromatic symmetric function $X_G(\mathbf{x})$.

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For a certain class of graphs the coefficients are symmetric functions.

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$$A_n^{(r)}(q, t) = ps(\omega X_{G_{n,r}}(\mathbf{x}, t))$$

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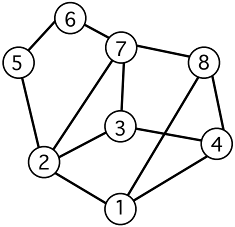
True for $1 \leq r \leq 2$ & $n - 2 \leq r \leq n$.

($r = 2$ follows from results of Procesi and Stanley)

Graph coloring



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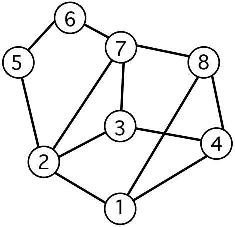

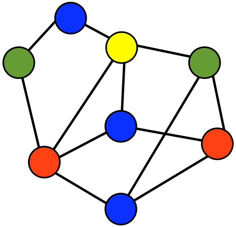

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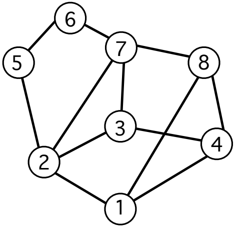
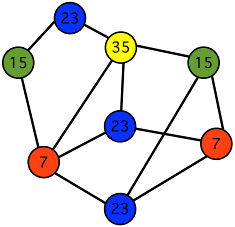
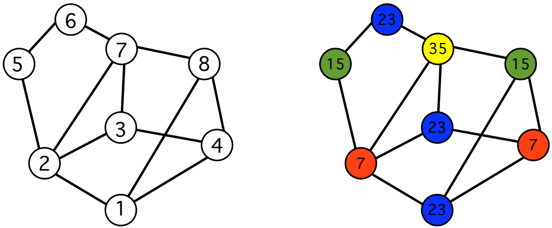




Let $V(G) = \{1, 2, \ldots, n\}$. Let $C(G)$ be set of proper colorings of G, where a proper coloring is a map $c : V(G) \to \mathbb{P}$ such that $c(i) \neq c(j)$ if $\{i, j\} \in E(G)$.

Graph coloring



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Chromatic symmetric function (Stanley, 1995)

$$X_G(\mathbf{x}) := \sum_{c \in C(G)} x_{c(1)} x_{c(2)} \cdots x_{c(n)}$$

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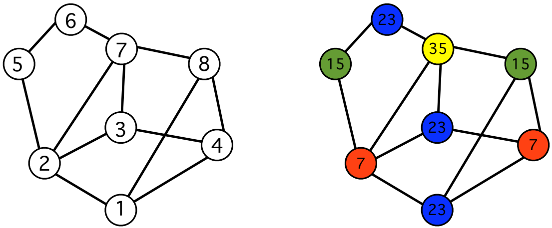

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$$X_G(\underbrace{1, 1, \ldots, 1}_{m}, 0, 0, \ldots) = \chi_G(m)$$

Stanley-Stembridge Conjecture

A symmetric function $f(\mathbf{x})$ is said to be *e-positive* if its expansion in the basis of elementary symmetric functions e_λ has nonnegative coefficients.

Conjecture (Stanley, Stembridge 1993)

If G is the incomparability graph of a $(3+1)$-free poset then X_G is e-positive.

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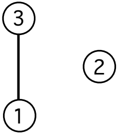
$P =$ 

$$X_G(\mathbf{x}) = 3e_3 + e_{2,1}$$

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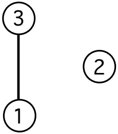
$P =$ 

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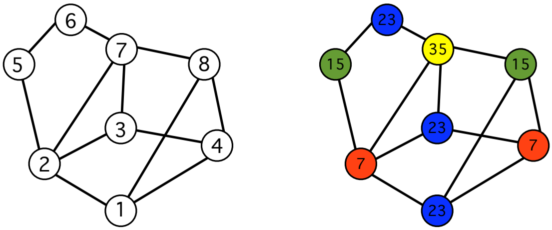
Theorem (Gasharov, 1996)

X_G is Schur-positive. The coefficient of s_λ is the number of P-tableaux of shape λ.

Chromatic quasisymmetric function



Chromatic quasisymmetric function (Shareshian and MW)

$$X_G(\mathbf{x}, t) := \sum_{c \in C(G)} t^{\mathrm{des}(c)} x_{c(1)} x_{c(2)} \cdots x_{c(n)}$$

where

$$\mathrm{des}(c) := |\{\{i,j\} \in E(G) : i < j \text{ and } c(i) > c(j)\}|.$$

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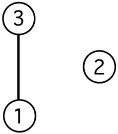
$P =$ 

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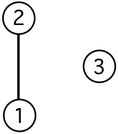
$P =$ 

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Let Λ be the ring of symmetric functions over $\mathbb{Z}$.
Unit interval orders are posets that are $(3+1)$-free and $(2+2)$-free.

Theorem (Shareshian and MW)

If G is the incomparability graph of a natural unit interval order then

- $X_G(\mathbf{x}, t) \in \Lambda[t]$
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A polynomial $f(\mathbf{x}, t) = \sum_{j=0}^{d} a_j(\mathbf{x}) t^j$ in $\Lambda[t]$ is

- e-positive if all the coefficients $a_j(\mathbf{x})$ are e-positive.
- e-unimodal if $a_j(\mathbf{x}) - a_{j-1}(\mathbf{x})$ is e-positive for all $1 \le j \le m$ and $a_j(\mathbf{x}) - a_{j+1}(\mathbf{x})$ is e-positive for all $m \le j \le d$

Refinement of Stanley-Stembridge Conjecture

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Let $G_{n,r}$ be the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{\{i, j\} \mid 0 < |j - i| < r\}$.

r	$X_{G_{n,r}}$
1	e_1^n
2	$\sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{\substack{k_1,\ldots,k_m \geq 2 \\ \sum k_i = n+1}} e_{k_1-1,k_2,\ldots,k_m} \; t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t$
$\vdots$	???
$n-2$	$e_n [n]_t [n-3]_t! [n-3]_t^2 + e_{n-1,1} \; t^{n-3} [n-4]_t! [n-2]_t ([n-3]_t + [2]_t [n-4]_t)$ $+ e_{n-2,2} t^{3n-10} [n-4]! [n-2]_t [2]_t$
$n-1$	$e_n [n]_t [n-2]_t! [n-2]_t + e_{n-1,1} t^{n-2} [n-2]_t!$
n	$e_n [n]_t!$

Chromatic quasisym. functions & q-generalized Eulerian

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Theorem (Shareshian and MW - refinement of Gasharov)

Let G be the incomparability graph of a natural unit interval order P. Then $X_G(\mathbf{x}, t)$ is Schur-positive. Moreover for each λ the coefficient of s_λ is

$$\sum_{T \in \mathcal{T}_{P,\lambda}} t^{\mathrm{inv}_G(T)}$$

where $\mathcal{T}_{P,\lambda}$ is the set of P-tableaux of shape λ.

Schur-unimodality is open.

Schur-unimodality $\Rightarrow$ unimodality conjecture for $A_n^{(r)}(q, t)$

Chromatic quasisym. functions & Hessenberg varieties

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$\mathcal{H}_{G_{n,r}} = \mathcal{H}_{n,r}$. True for $1 \le r \le 2$ & $n-2 \le r \le n$.
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- $\Rightarrow$ Schur-unimodality conjecture for $X_G(\mathbf{x},t)$
- $\Rightarrow$ q-unimodality conjecture for $A_G(q,t)$ (includes $A_n^{(r)}(q,t)$)
- $\Rightarrow$ multiplicity of irreducibles in Tymoczko's representation.

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Conjecture

Tymoczko's representation on $H^{2j}(\mathcal{H}_G)$ is a permutation representation in which each point stabilizer is a Young subgroup.

- $\Rightarrow$ Stanley-Stembridge e-positivity conjecture for unit interval orders.