The Geometry of Lecture Hall Partitions and Quadratic Permutation Statistics

Katie L. Bright
Carla D. Savage
Dept. of Computer Science
North Carolina State University

Lecture Hall Partitions L_n

[Bousquet-Mélou & Eriksson 1997]

integer seqs. $(\lambda_1, \lambda_2, \dots, \lambda_n)$ satisf.

$$0 \le \frac{\lambda_1}{1} \le \frac{\lambda_2}{2} \le \dots \le \frac{\lambda_n}{n}$$

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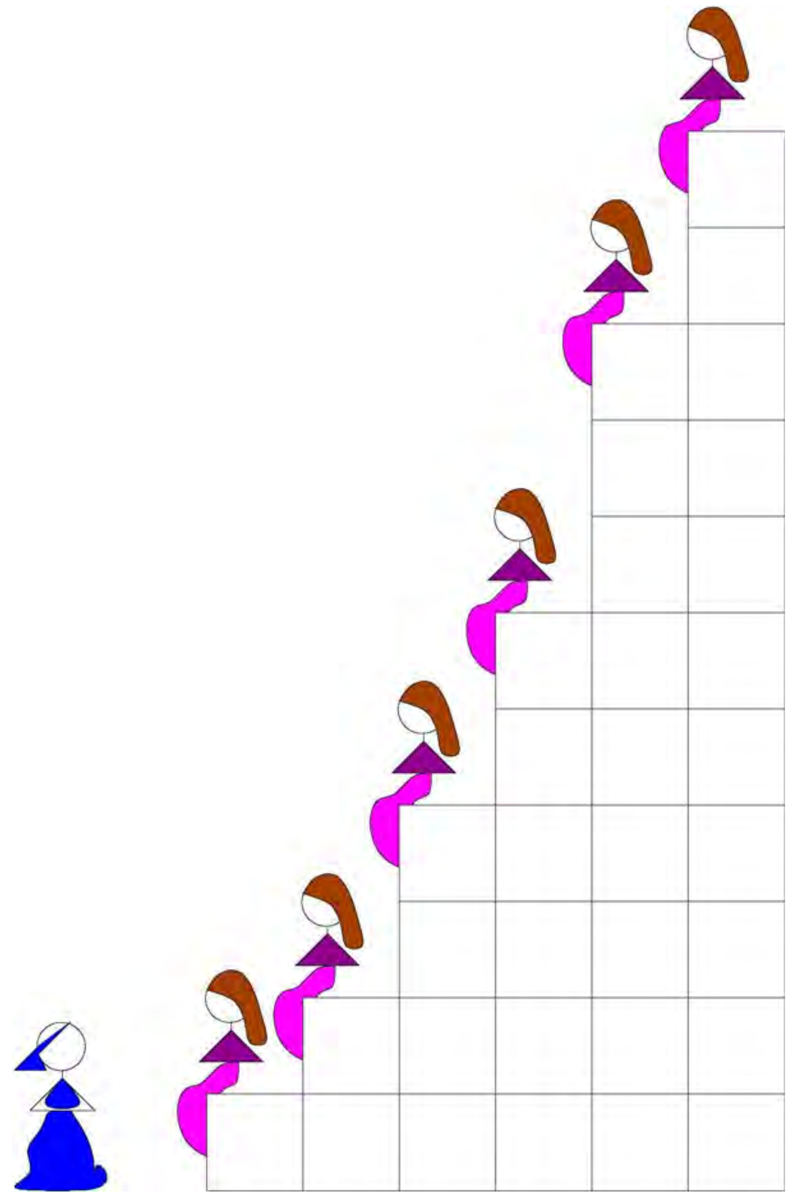

$\lambda = (1, 2, 4, 6, 9, 11) \in L_6$

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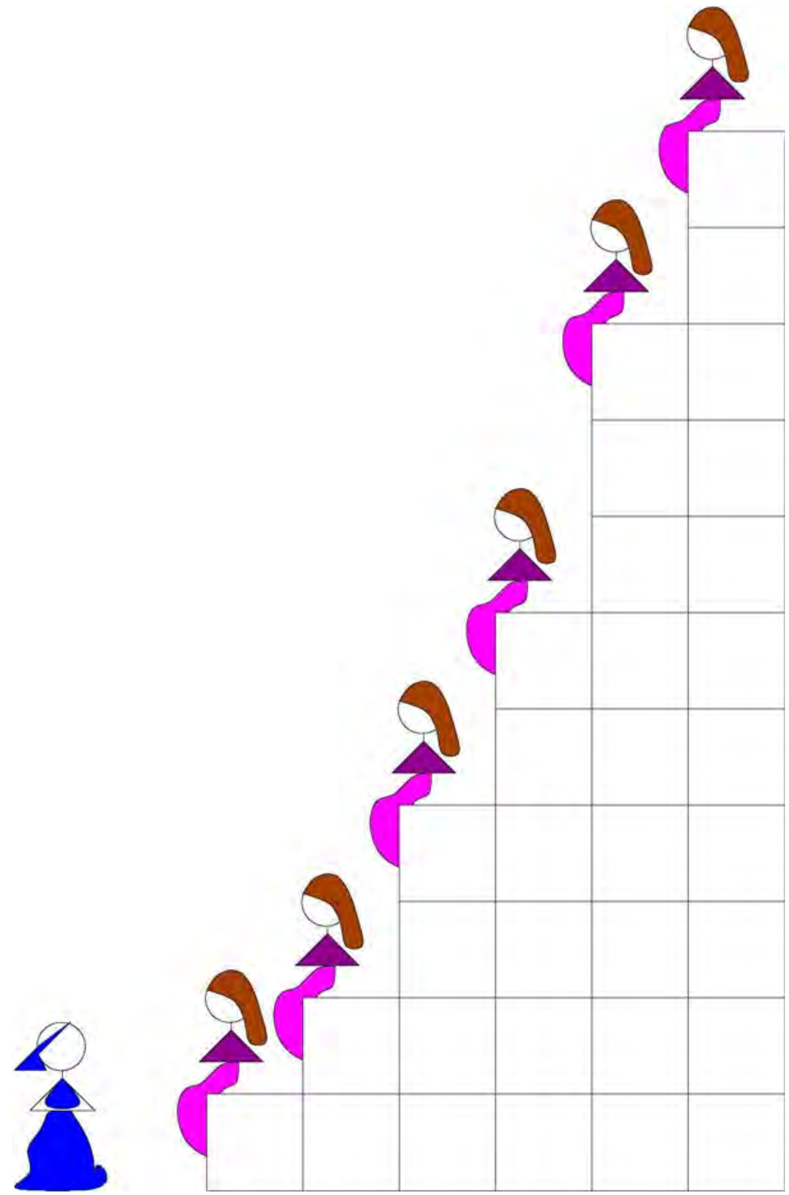
Thm. [BME1 1997]

$$\sum_{\lambda \in L_n} q^{|\lambda_1+\lambda_2+\cdots+\lambda_n|} = \prod_{i=1}^{n} \frac{1}{1-q^{2i-1}}$$



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finite form of Euler's "odd = distinct" partition theorem

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Anti-lecture Hall Compositions A_n

[Corteel, S. 2003]

integer seqs. $(\lambda_1, \lambda_2, \dots, \lambda_n)$ satisf.

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \dots \geq \frac{\lambda_n}{n} \geq 0$$

Thm. [CS 2003]

$$\sum_{\lambda \in A_n} q^{|\lambda_1 + \lambda_2 + \dots + \lambda_n|} = \prod_{i=1}^{n} \frac{1+q^i}{1-q^{i+1}}$$

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Thm. [BME2 1999] (refined LHPT)

$$\sum_{\lambda \in L_n} q^{|\lambda|} u^{|\lceil \lambda \rceil|} = \prod_{i=1}^{n} \frac{1+uq^i}{1-u^2 q^{n+i}}$$

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Implied by this: $L_n(u,q) = A_n(u q^{n+1}, q^{-1})$

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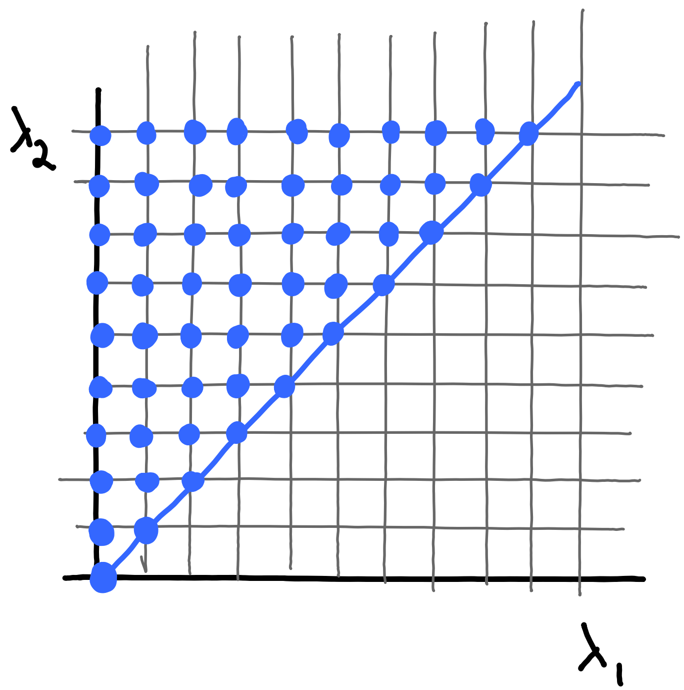
How to understand / explain this

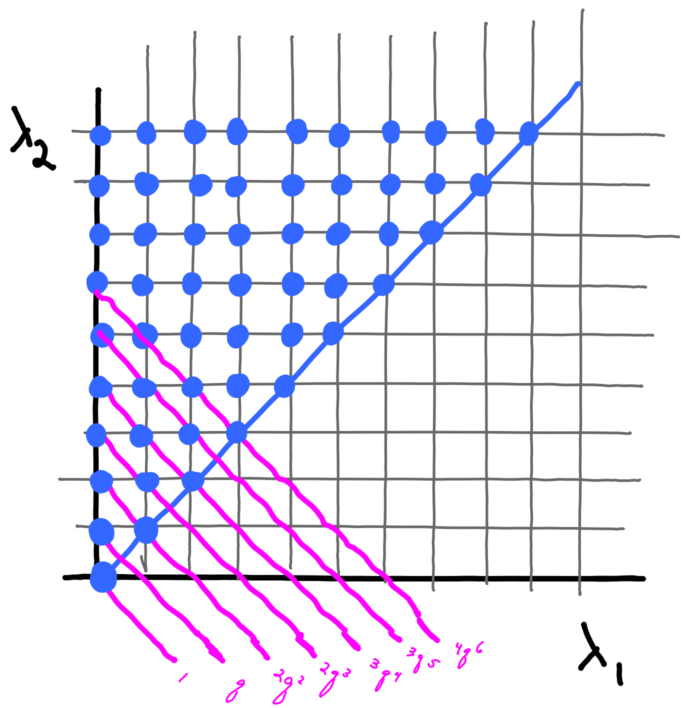
Combinatorially?

Integer Partitions



P_n: $\quad 0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$

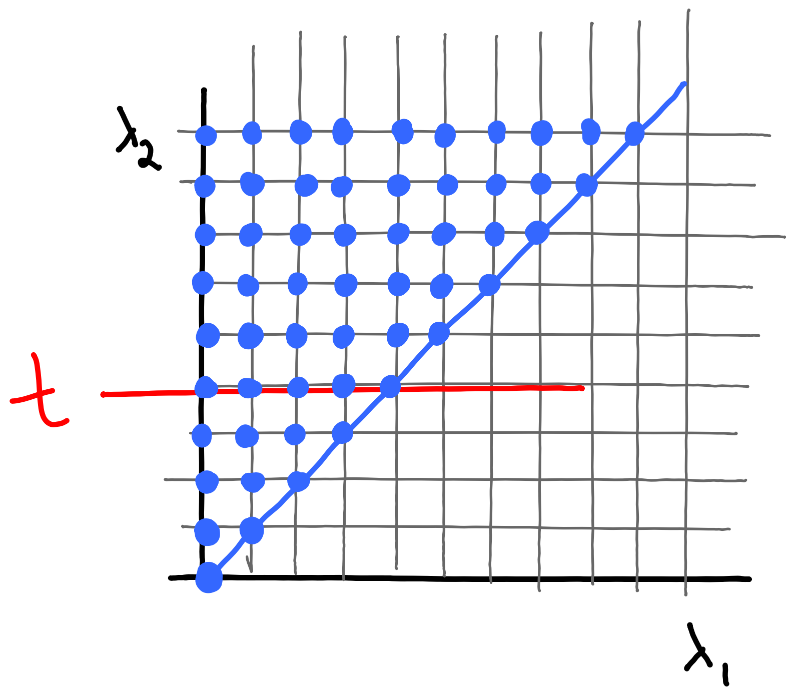
Integer Partitions



$$q^{|\lambda_1+\lambda_2+\ldots+\lambda_n|}$$

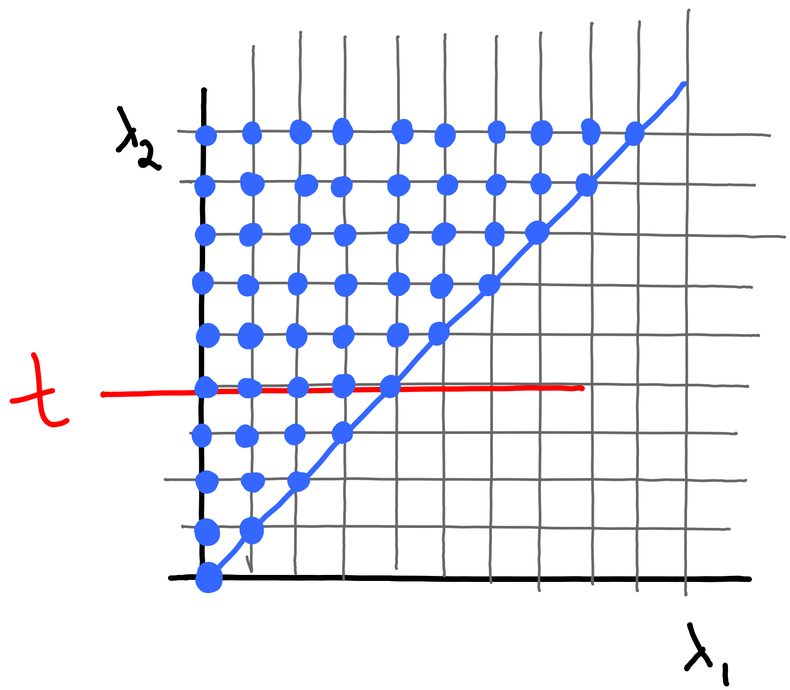
P_n : $\quad 0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$

"Partitions in a box" ($n \times t$)



$P_n^{(t)}$: $0 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_n \le t$

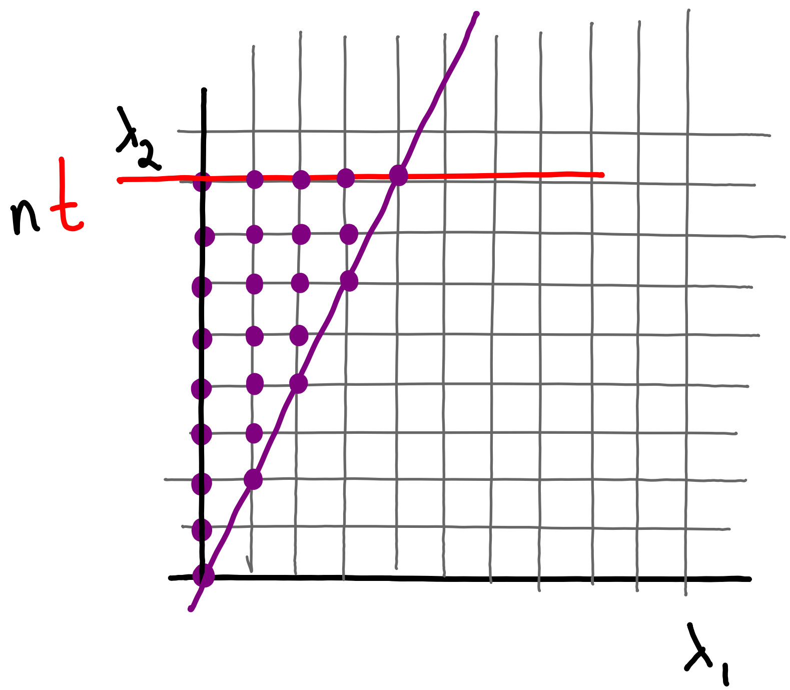
"Partitions in a box" $(n \times t)$



$$|P_n^{(t)}| = \binom{n+t}{t}$$

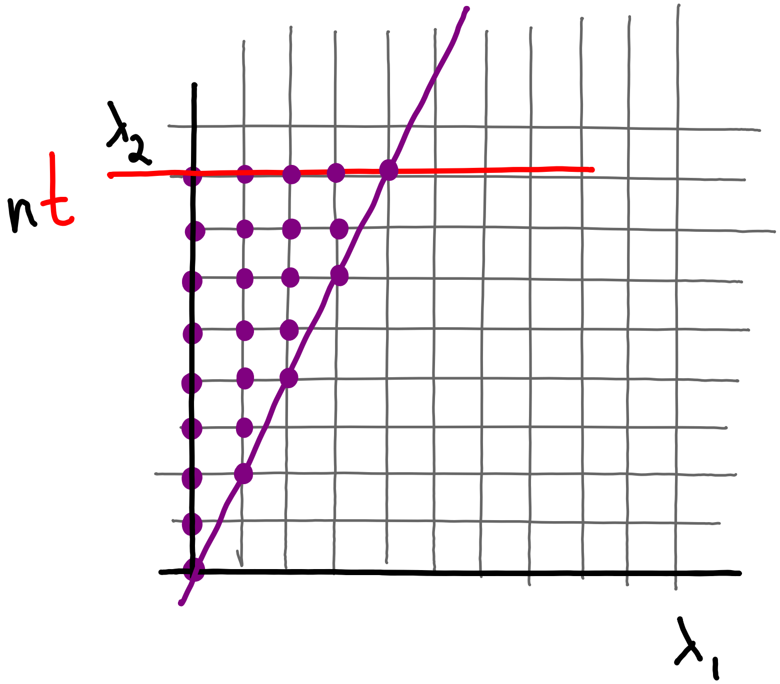
$$P_n^{(t)} : \quad 0 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_n \le t$$

"Lecture Hall Partitions in a box" $(n \times nt)$



$L_n^{(t)}$: $\quad 0 \le \frac{\lambda_1}{1} \le \frac{\lambda_2}{2} \le \cdots \le \frac{\lambda_n}{n} \le t$

"Lecture Hall Partitions in a box" $(n \times nt)$

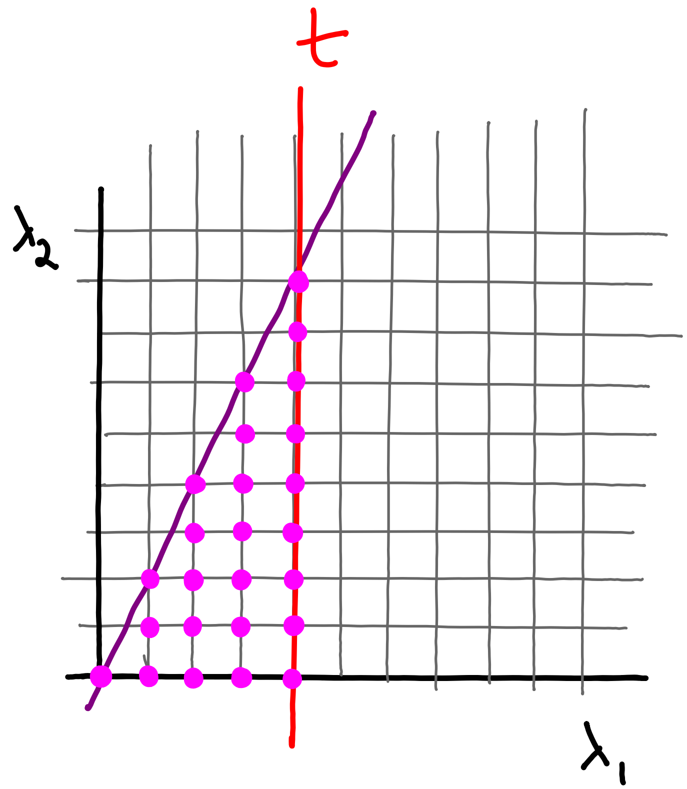


Thm. [Corteel, Lee, S. 2005]

$$\left| L_n^{(t)} \right| = (t+1)^n$$

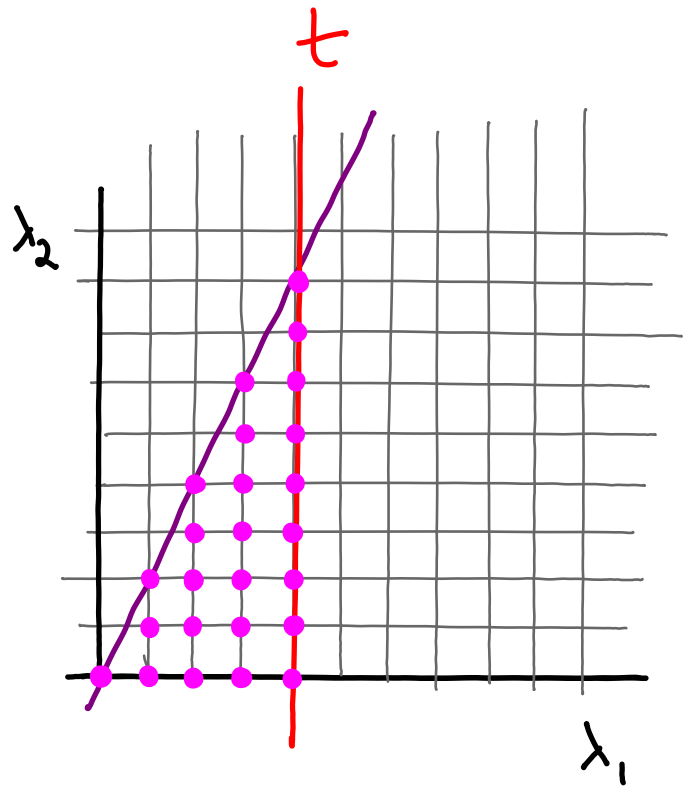
$L_n^{(t)}$: $\quad 0 \le \frac{\lambda_1}{1} \le \frac{\lambda_2}{2} \le \cdots \le \frac{\lambda_n}{n} \le t$

"Anti-lecture Hall Compositions in a box" $(n \times t)$



$$A_n^{(t)} : \quad t \geq \frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \cdots \geq \frac{\lambda_n}{n} \geq 0$$

"Anti-lecture Hall Compositions in a box" $(n \times t)$



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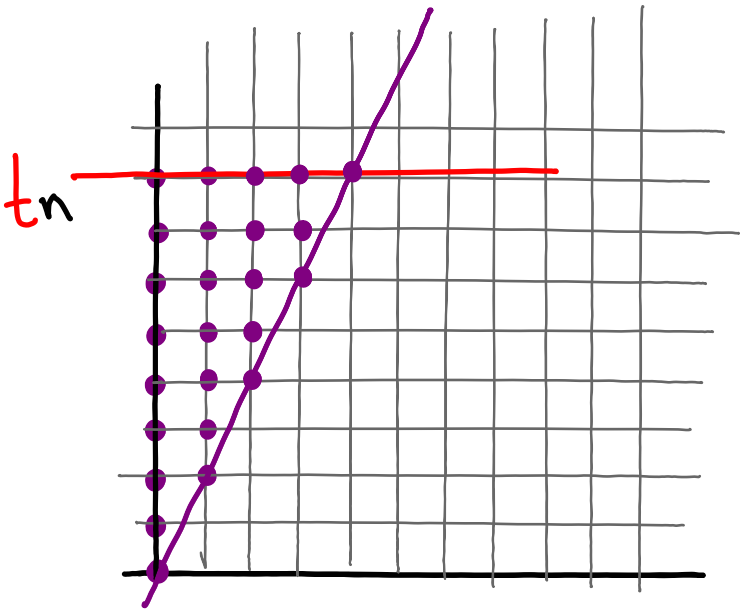
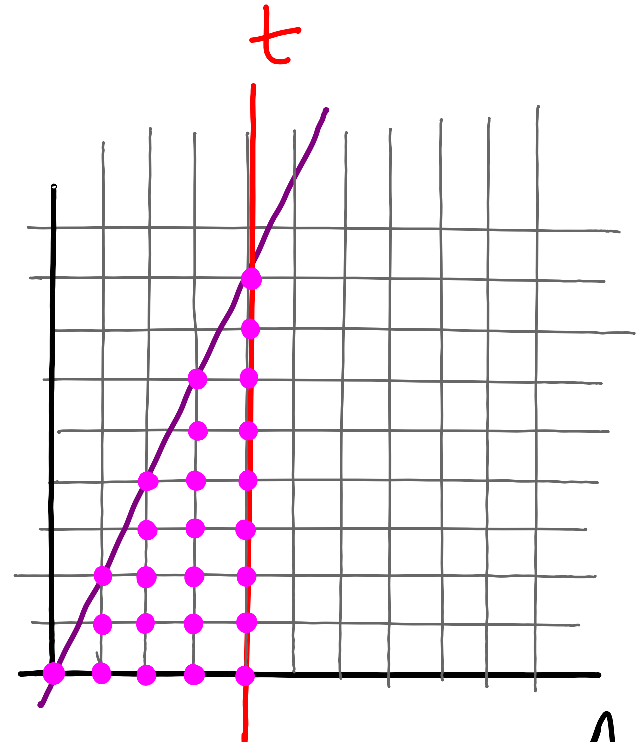


$L_n^{(t)}$

Can map

$L_n^{(t)} \to A_n^{(t)}$

bijectively ?



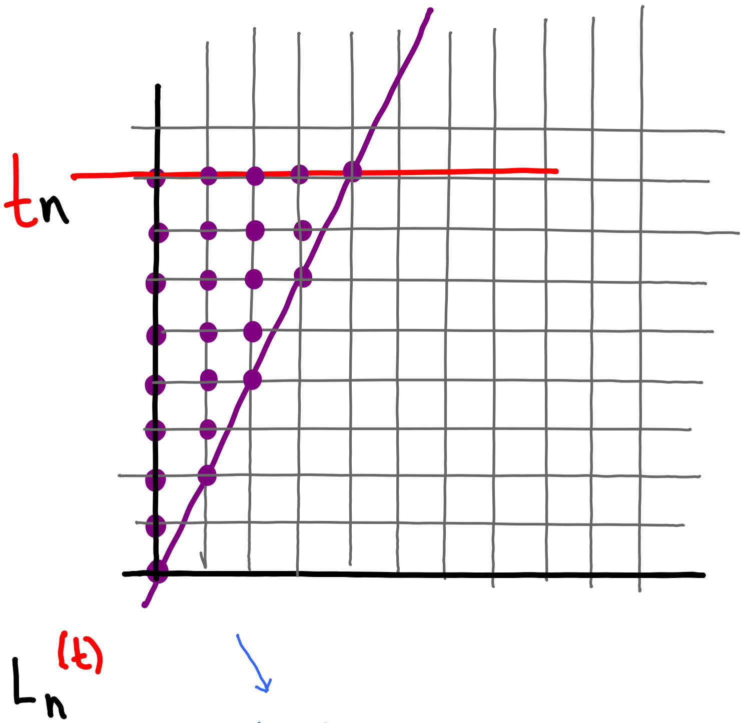
$A_n^{(t)}$

Can map

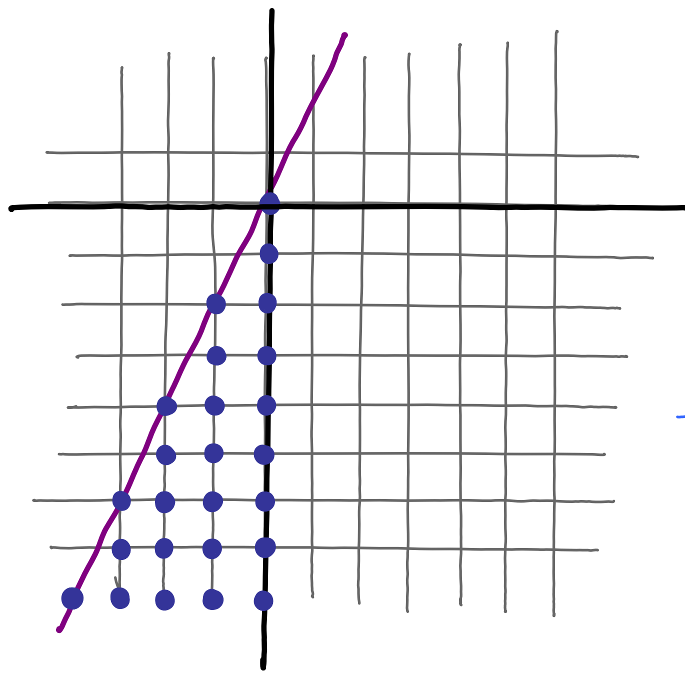
$L_n^{(t)} \to A_n^{(t)}$

bijectively via

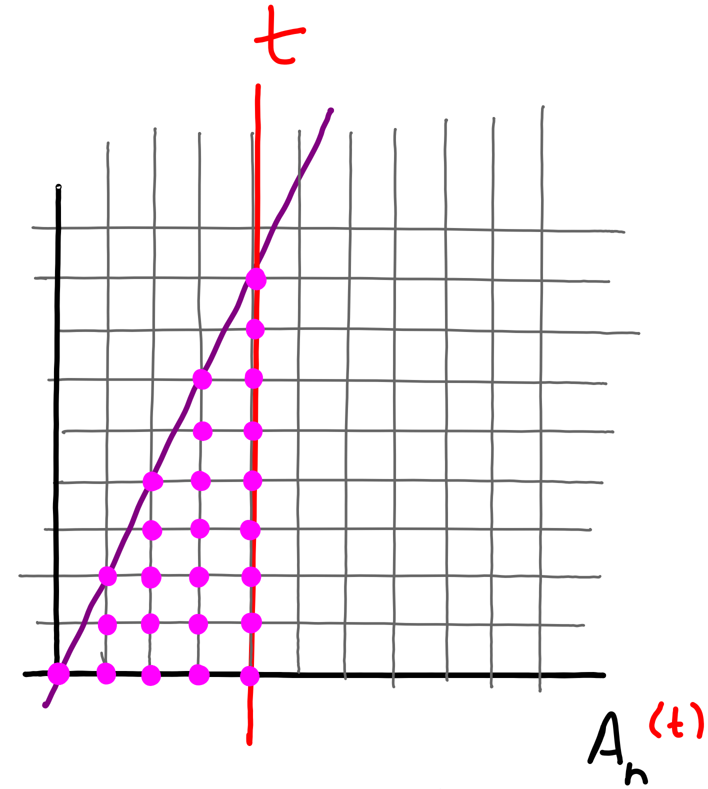
reciprocity



$(i,j) \to (-i,-j) \to$



$\to (i,j) \to (i+t,\ j+nt) \to$



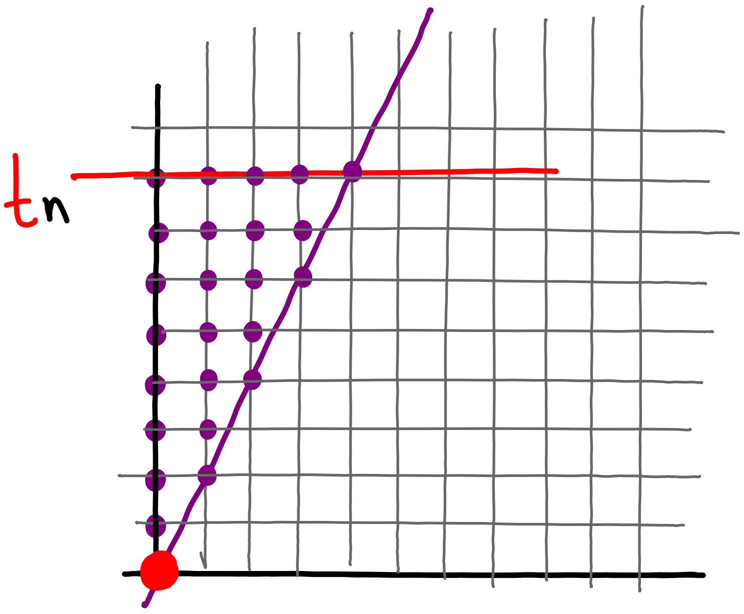
Can map

$L_n^{(t)} \to A_n^{(t)}$

bijectively via

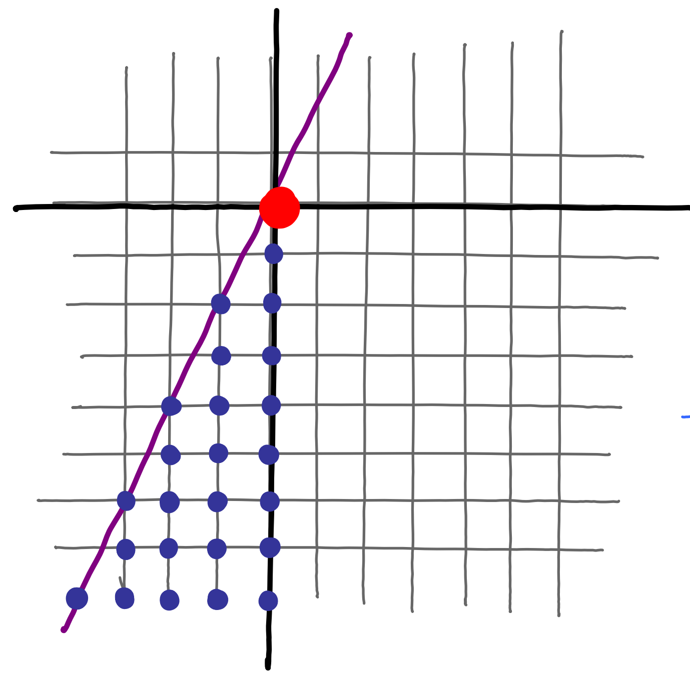
reciprocity

(But map depends on t)



$L_n^{(t)}$

$(i,j) \to (-i,-j) \to$



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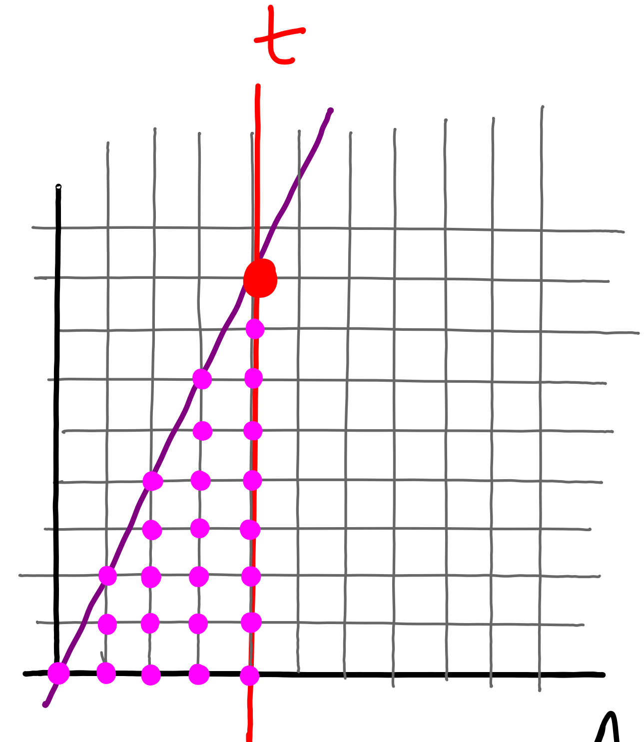
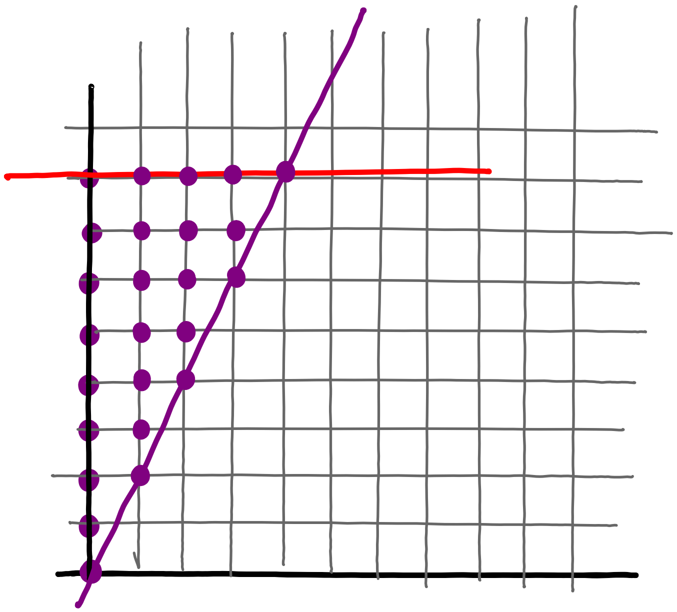
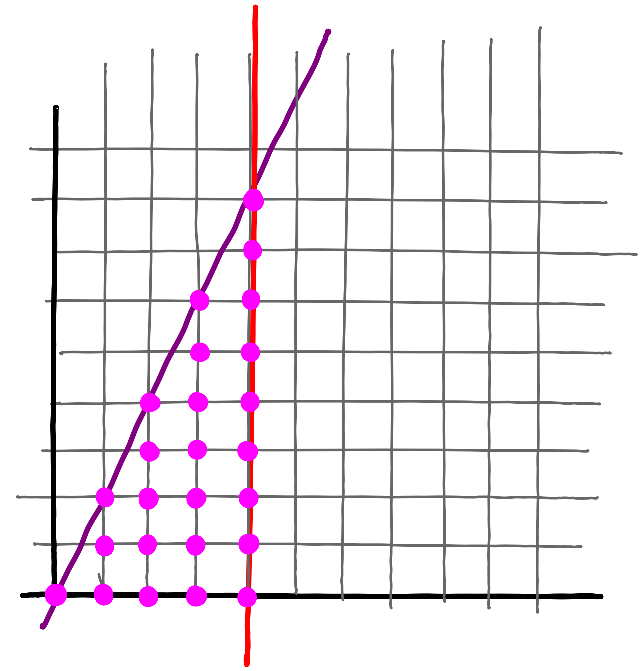


$A_n^{(t)}$



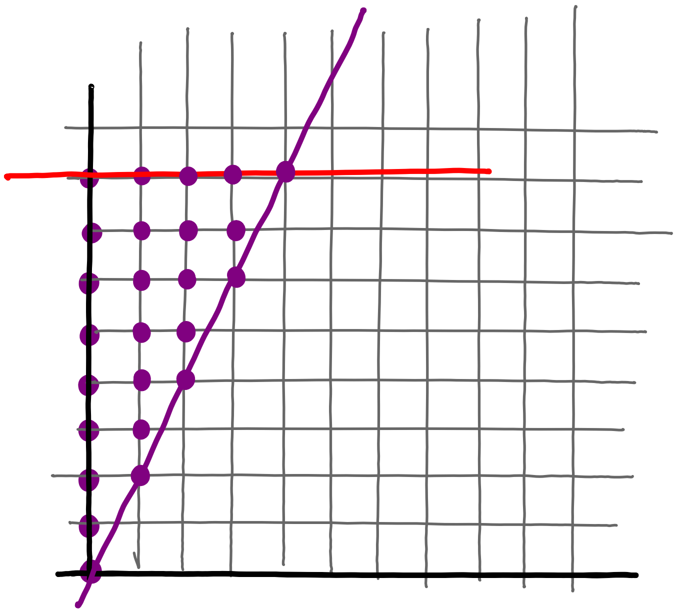
$L_n^{(t)}$



$A_n^{(t)}$

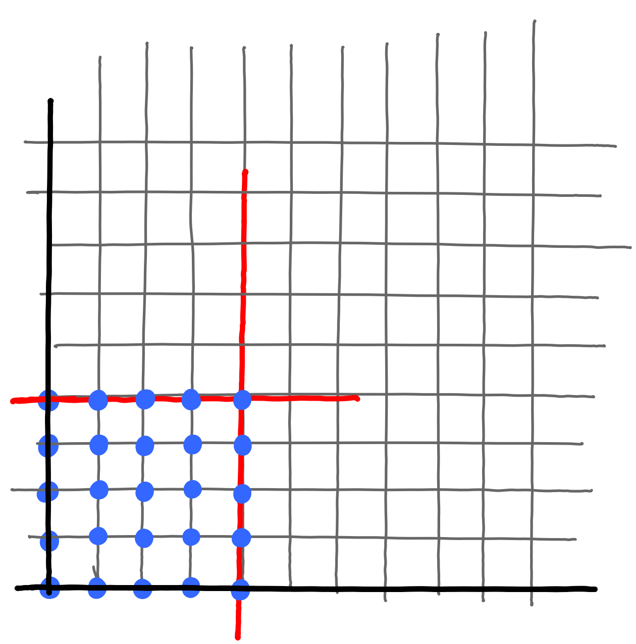
- Is there a bijection $L_n \to A_n$ that maps $L_n^{(t)}$ to $A_n^{(t)}$?

- Can it explain why $L_n(u,q) = A_n(uq^{n+1}, q^{-1})$?



$L_n^{(t)}$

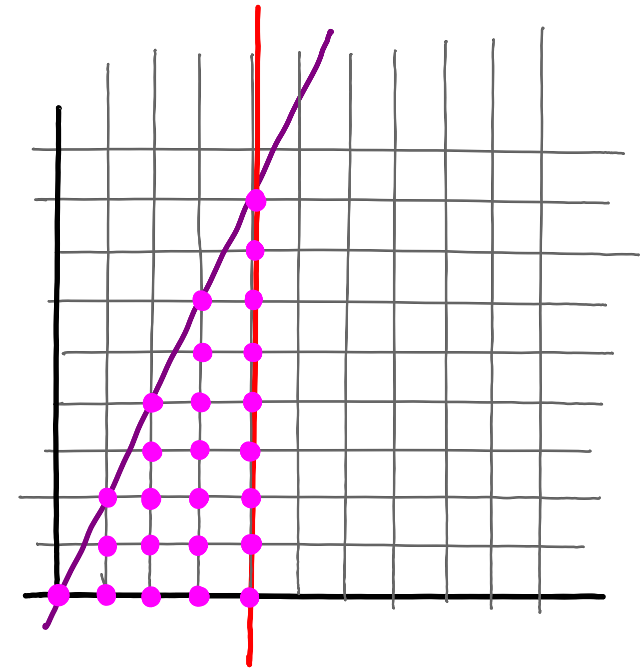
$0 \le \frac{\lambda_1}{1} \le \frac{\lambda_2}{2} \le \cdots \frac{\lambda_n}{n} \le t$



$Q_n^{(t)}$

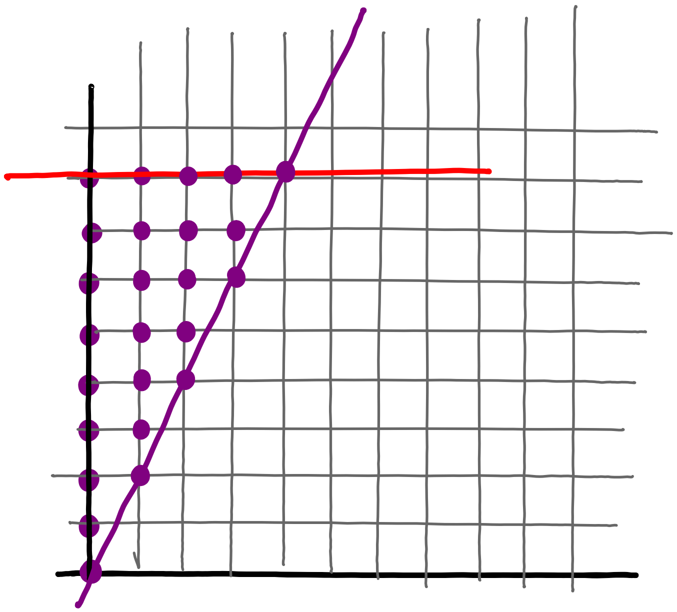
$0 \le \lambda_i \le t$

$1 \le i \le n$



$A_n^{(t)}$

$t \ge \frac{\lambda_1}{1} \ge \frac{\lambda_2}{2} \ge \cdots \frac{\lambda_n}{n} \ge 0$



$$|L_n^{(t)}| = (t+1)^n$$

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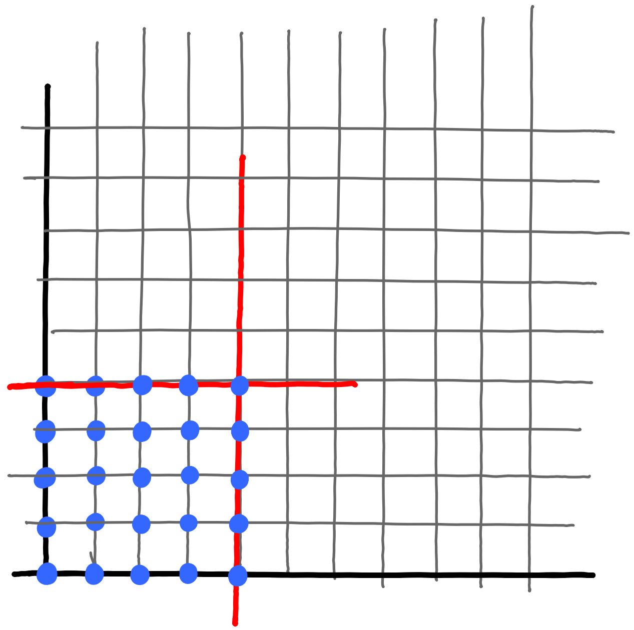
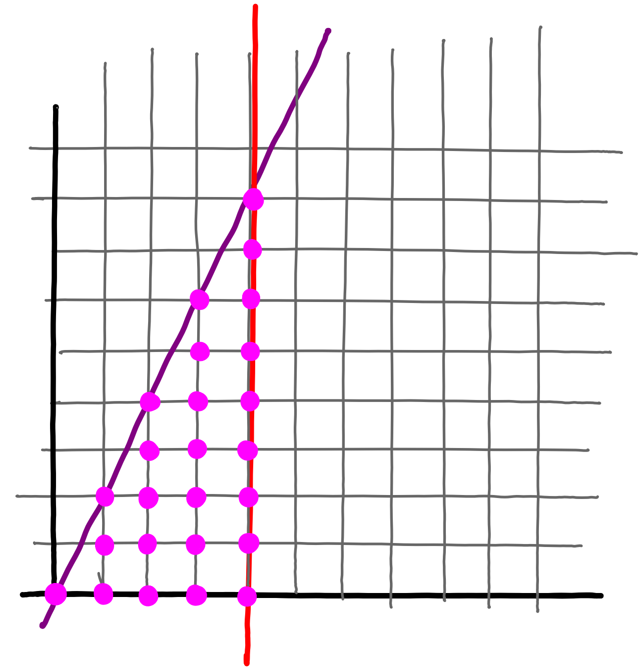


$$|Q_n^{(t)}| = (t+1)^n$$

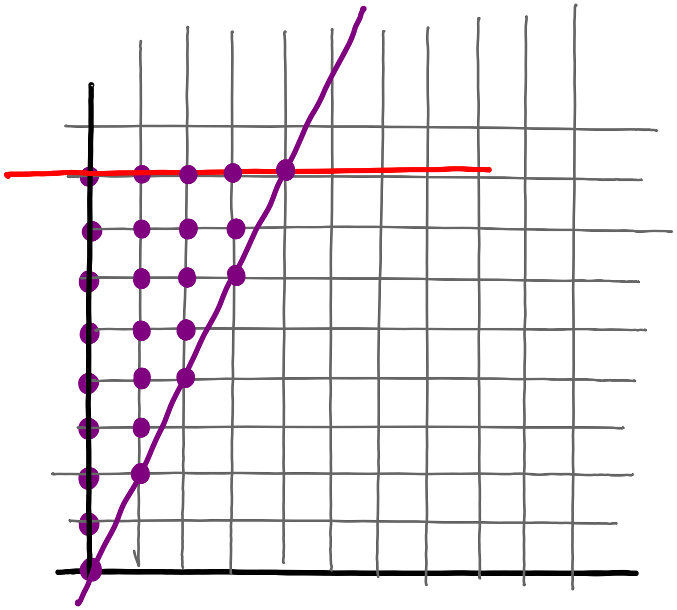
$$0 \le \lambda_i \le t$$

$$1 \le i \le n$$



$$|A_n^{(t)}| = (t+1)^n$$

$$t \ge \frac{\lambda_1}{1} \ge \frac{\lambda_2}{2} \ge \dots \frac{\lambda_n}{n} \ge 0$$



$$\left|L_n^{(t)}\right| = (t+1)^n$$

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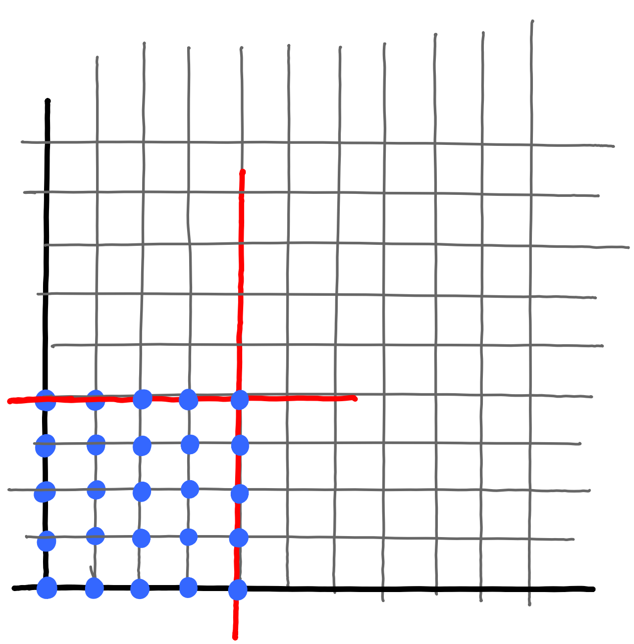

$$\left|Q_n^{(t)}\right| = (t+1)^n$$

$$0 \le \lambda_i \le t$$

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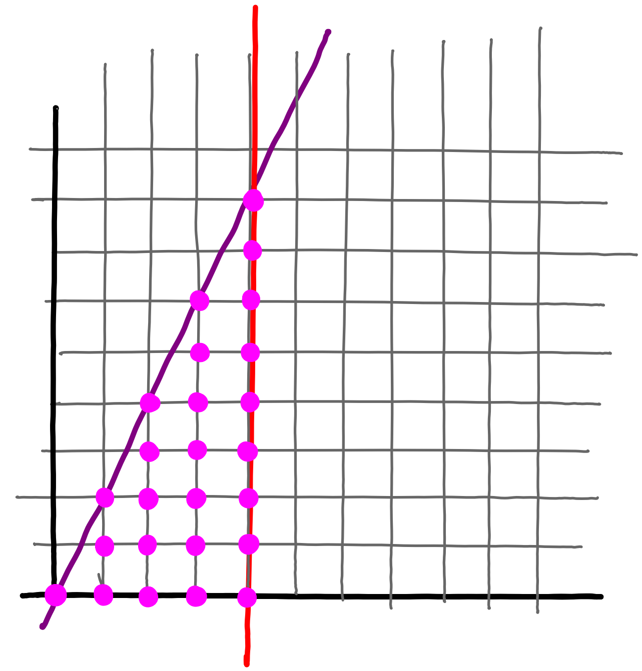


$$\left|A_n^{(t)}\right| = (t+1)^n$$

$$t \ge \frac{\lambda_1}{1} \ge \frac{\lambda_2}{2} \ge \cdots \frac{\lambda_n}{n} \ge 0$$

"Why?" - Matt Beck

Bijection $\Theta : \mathbb{Z}^n_{\geq 0} \longrightarrow L_n$ for Lecture Hall Partitions

Bijection $\Theta : \mathbb{Z}^n_{\geq 0} \longrightarrow L_n$ for Lecture Hall Partitions

$\boxed{\text{EXAMPLE}}$

Given $p = (9, 0, 3, 3, 5, 4, 3, 8, 1, 8, 2, 9) \in \mathbb{Z}^{12}_{\geq 0}$

Bijection $\Theta: \mathbb{Z}_{\geq 0}^n \longrightarrow L_n$ for Lecture Hall Partitions

Given $\quad p = (9, 0, 3, 3, 5, 4, 3, 8, 1, 8, 2, 9) \in \mathbb{Z}_{\geq 0}^{12}$

Sort p $\quad w = (0, 1, 2, 3, 3, 3, 4, 5, 8, 8, 9, 9)$

Bijection $\Theta : \mathbb{Z}^n_{\geq 0} \longrightarrow L_n$ for Lecture Hall Partitions

Given $\quad p = (9, 0, 3, 3, 5, 4, 3, 8, 1, 8, 2, 9) \in \mathbb{Z}^{12}_{\geq 0}$

etc.

Sort p stably $\quad w = (0, 1, 2, 3, 3, 3, 4, 5, 8, 8, 9, 9)$

Bijection $\Theta : \mathbb{Z}^n_{\geq 0} \to L_n$ for Lecture Hall Partitions

Given $p = (9, 0, 3, 3, 5, 4, 3, 8, 1, 8, 2, 9) \in \mathbb{Z}^{12}_{\geq 0}$

etc.

Sort p stably $\quad w = (0, 1, 2, 3, 3, 3, 4, 5, 8, 8, 9, 9)$

Let $\epsilon_i = \left| \{ j \mid i < j \text{ and } w_i > p_j \} \right|$

$\epsilon = (0, 0, 0, 2, 2, 2, 3, 4, 2, 1, 10, 0)$

Bijection $\Theta : \mathbb{Z}^n_{\geq 0} \longrightarrow L_n$ for Lecture Hall Partitions

Given $\quad p = (9, 0, 3, 3, 5, 4, 3, 8, 1, 8, 2, 9) \in \mathbb{Z}^{12}_{\geq 0}$

etc.

Sort p stably $\quad w = (0, 1, 2, 3, 3, 3, 4, 5, 8, 8, 9, 9)$

$\epsilon = (0, 0, 0, 2, 2, 2, 3, 4, 2, 1, 10, 0)$

Bijection $\Theta: \mathbb{Z}^n_{\geq 0} \longrightarrow L_n$ for Lecture Hall Partitions

Given $p = (9, 0, 3, 3, 5, 4, 3, 8, 1, 8, 2, 9) \in \mathbb{Z}^{12}_{\geq 0}$

etc.

Sort p stably $\quad w = (0, 1, 2, 3, 3, 3, 4, 5, 8, 8, 9, 9)$

$w_i \to i\, w_i$: $\quad (0, 2, 6, 12, 15, 18, 28, 40, 72, 80, 99, 108)$

$\epsilon = (0, 0, 0, 2, 2, 2, 3, 4, 2, 1, 10, 0)$

Bijection $\Theta : \mathbb{Z}^n_{\geq 0} \to L_n$ for Lecture Hall Partitions

Given $p = (9, 0, 3, 3, 5, 4, 3, 8, 1, 8, 2, 9) \in \mathbb{Z}^{12}_{\geq 0}$

etc.

Sort p stably $\quad w = (0, 1, 2, 3, 3, 3, 4, 5, 8, 8, 9, 9)$

$w_i \to i w_i$: $\quad (0, 2, 6, 12, 15, 18, 28, 40, 72, 80, 99, 108)$

$\epsilon = (0, 0, 0, 2, 2, 2, 3, 4, 2, 1, 10, 0)$

$i w_i - \epsilon_i$:

$\lambda = (0, 2, 6, 10, 13, 16, 25, 36, 70, 79, 89, 108)$

Define for $\pi = \pi_1 \pi_2 \ldots \pi_n \in S_n$

the inversion sequence of π

$$\epsilon(\pi) = (\epsilon_1, \ldots, \epsilon_n), \quad \text{where}$$

ϵ_i = # of elements to the right of i in π which are smaller than i

(In the example, if π stably sorts p, then $\epsilon = \epsilon(\pi^{-1})$)

Well-known: $\pi \to \epsilon(\pi)$ is a bijection $S_n \to I_n$,

$$I_n = \{(e_1, \ldots, e_n) \mid 0 \le e_i < i\}$$

Bijection $\Theta : \mathbb{Z}^n_{\geq 0} \longrightarrow L_n$ for Lecture hall partitions

Given $p \in \mathbb{Z}^n_{\geq 0}$

1. Let π^{-1} be the unique permutation that stably sorts p into w
2. Let ϵ be the inversion sequence of π

Then $\Theta(p) = (\lambda_1, \dots, \lambda_n)$ where $\lambda_i = i w_i - \epsilon_i$ $\quad i = 1, 2, \dots, n$

Theorem [Bright, S 2009]

$\Theta : \mathbb{Z}^n_{\geq 0} \to L_n$ is a bijection that restricts

to a bijection $Q_n^{(t)} \to L_n^{(t)}$.

Bijection $\Phi : \mathbb{Z}^n_{\geq 0} \longrightarrow A_n$ for Anti-lecture hall compositions

Given $p \in \mathbb{Z}^n_{\geq 0}$

1. Let π^{-1} be the unique permutation that stably reverse sorts p into w
2. Let ϵ be the inversion sequence of π

Then $\Phi(p) = (\lambda_1, \ldots, \lambda_n)$ where $\lambda_i = i w_i + \epsilon_i$ $\quad i = 1, 2, \ldots, n$

Theorem [Bright, S 2009]

$\Phi : \mathbb{Z}^n_{\geq 0} \to A_n$ is a bijection that restricts

to a bijection $Q_n^{(t)} \to A_n^{(t)}$.

Proofs rely on :

If $w_i = w_{i+1}$ then $\epsilon_i \geq \epsilon_{i+1}$

for LHP:

$$\frac{\lambda_i - \epsilon_i}{i} \leq \frac{\lambda_{i+1} - \epsilon_{i+1}}{i+1}$$

for ALHC :

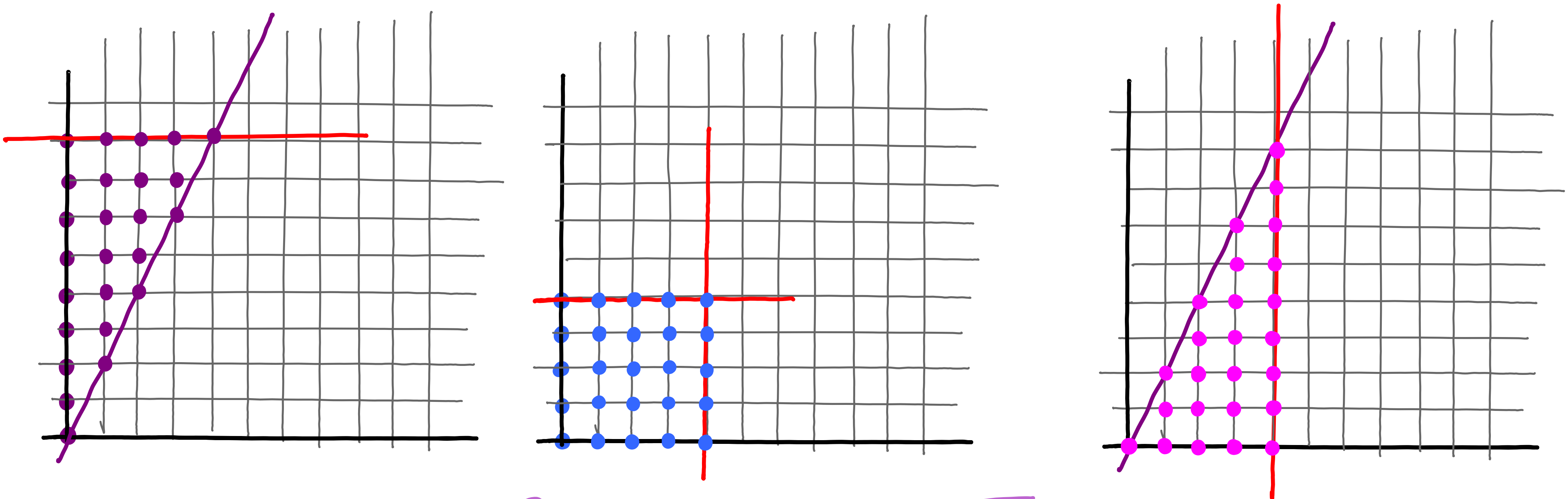
$$\frac{\lambda_i + \epsilon_i}{i} \geq \frac{\lambda_{i+1} + \epsilon_{i+1}}{i+1}$$

Proofs rely on :

If $w_i = w_{i+1}$ then $\epsilon_i \geq \epsilon_{i+1}$

and :

Lemma Given $w = (w_1 \leq w_2 \ldots \leq w_n)$ or $w = (w_1 \geq w_2 \geq \ldots \geq w_n)$, the bijection $\pi \to \epsilon(\pi)$ restricts to a bijection between (distinguishable) permutations $\pi(w)$ of w and sequences in I_n satisfying $\epsilon_i \geq \epsilon_{i+1}$ if $w_i = w_{i+1}$.

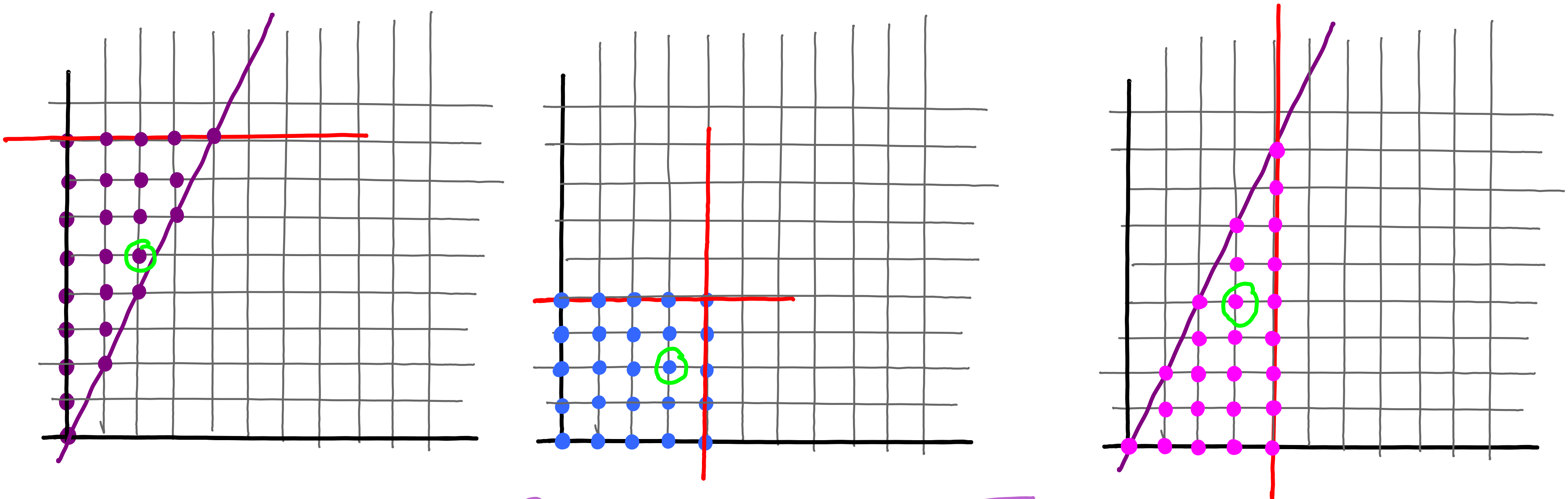


$$L_n^{(t)} \xleftarrow{\Theta} Q_n^{(t)} \xrightarrow{\Phi} A_n^{(t)}$$

$$0 \le \frac{\lambda_1}{1} \le \frac{\lambda_2}{2} \le \dots \frac{\lambda_n}{n} \le t$$

$$0 \le \lambda_i \le t$$
$$1 \le i \le n$$

$$t \ge \frac{\lambda_1}{1} \ge \frac{\lambda_2}{2} \ge \dots \frac{\lambda_n}{n} \ge 0$$



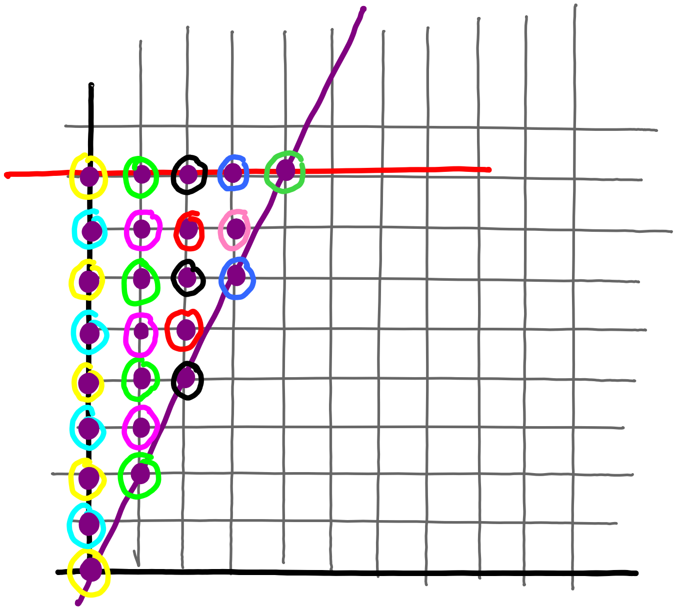
$$L_n^{(t)} \xleftarrow{\;\Theta\;} Q_n^{(t)} \xrightarrow{\;\Phi\;} A_n^{(t)}$$

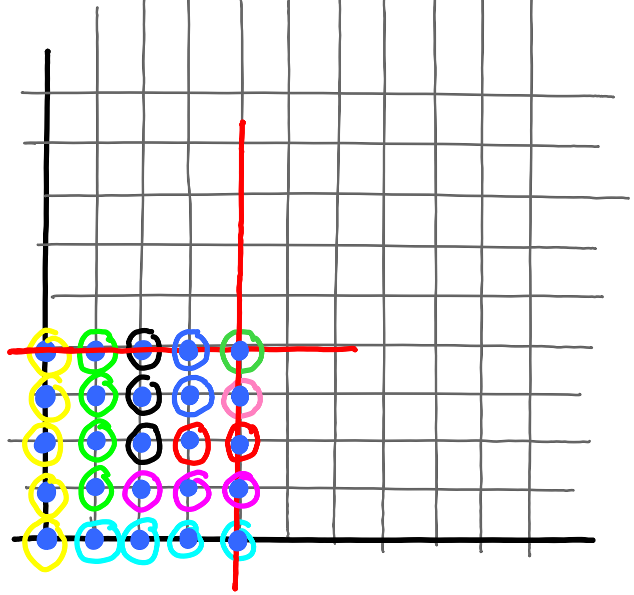
$0 \le \frac{\lambda_1}{1} \le \frac{\lambda_2}{2} \le \cdots \frac{\lambda_n}{n} \le t$

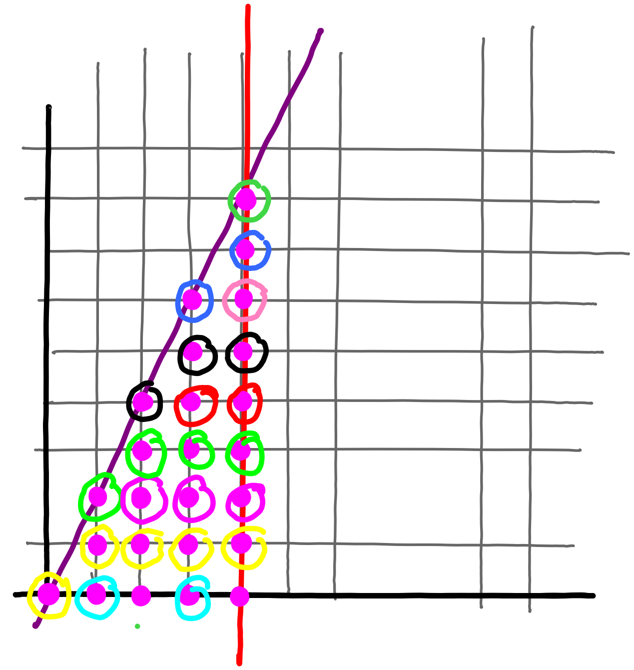
$0 \le \lambda_i \le t$

$1 \le i \le n$

$t \ge \frac{\lambda_1}{1} \ge \frac{\lambda_2}{2} \ge \cdots \frac{\lambda_n}{n} \ge 0$







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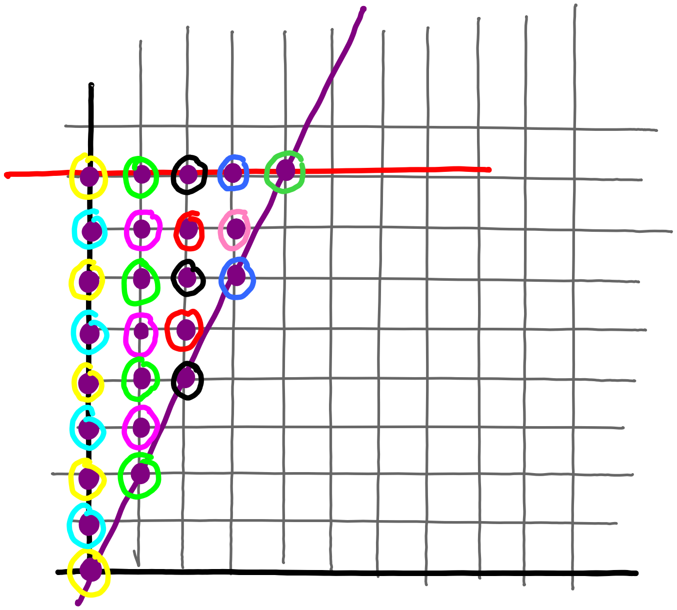

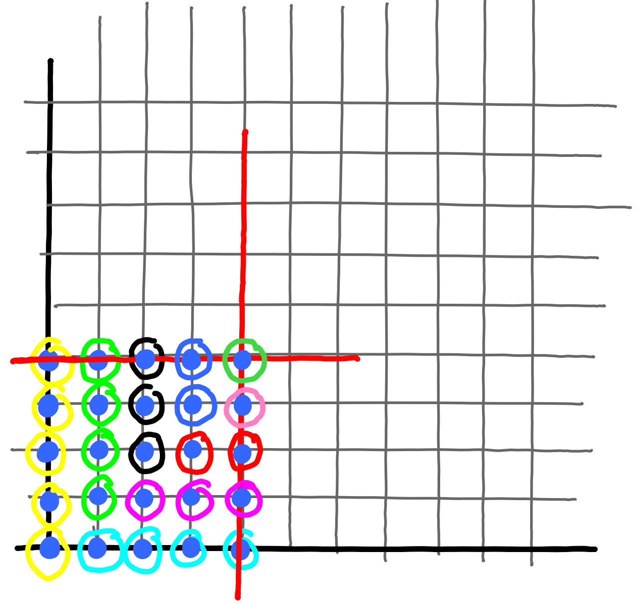

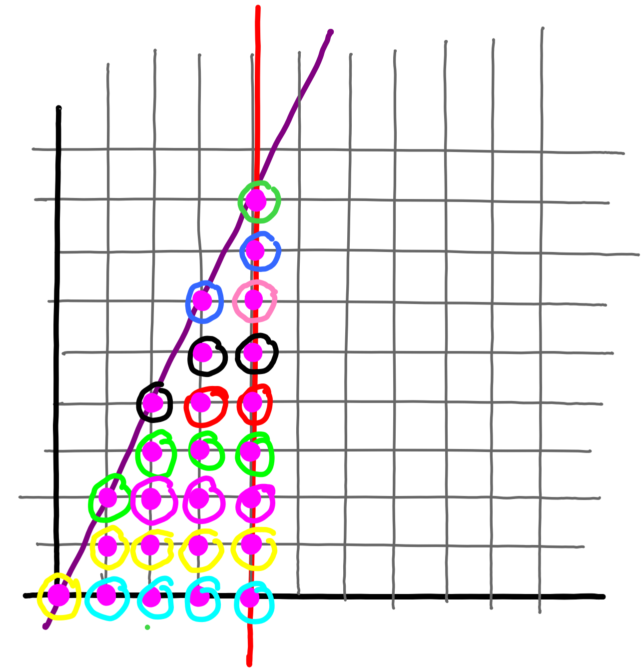

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$t \geq \frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \dots \frac{\lambda_n}{n} \geq 0$

Now: $\Phi\Theta^{-1} : L_n \longrightarrow A_n$

But ...?

Def. For $\pi \in S_n$

$$\operatorname{Inv}(\pi) = \left| \{ (i,j) \mid i < j \text{ and } \pi_i > \pi_j \} \right|$$

Note: $$\operatorname{Inv}(\pi) = \sum_{i=1}^{n} \epsilon_i(\pi)$$

Lemma: For $\pi \in S_n$

$$\operatorname{Inv}(\pi) = \operatorname{Inv}(\pi^{-1})$$

(even though $\epsilon(\pi) \neq \epsilon(\pi^{-1})$)

Lemma

If α stably sorts $p_1 p_2 \ldots p_n$ into $w_1 \le w_2 \le \ldots \le w_n$

and β stably sorts $p_n p_{n-1} \ldots p_1$ into $w_1 \ge w_2 \ge \ldots \ge w_n$

then

$$\mathrm{inv}(\alpha) = \mathrm{inv}(\beta).$$

Define

$$L_n(u,q) = \sum_{\lambda \in L_n} u^{|\lceil \lambda \rceil|} q^{|\lambda|}$$

$$A_n(u,q) = \sum_{\lambda \in A_n} u^{|\lfloor \lambda \rfloor|} q^{|\lambda|}$$

$$\lceil \lambda \rceil = \left(\left\lceil \frac{\lambda_1}{1} \right\rceil, \left\lceil \frac{\lambda_2}{2} \right\rceil, \ldots, \left\lceil \frac{\lambda_n}{n} \right\rceil \right) \qquad \lfloor \lambda \rfloor = \left(\left\lfloor \frac{\lambda_1}{1} \right\rfloor, \left\lfloor \frac{\lambda_2}{2} \right\rfloor, \ldots, \left\lfloor \frac{\lambda_n}{n} \right\rfloor \right.$$

$w_1 \le w_2 \le \ldots \le w_n$ $\qquad$ $w_1 \ge w_2 \ge \ldots \ge w_n$

Thm $\quad L_n(u,q) = A_n\left(uq^{n+1}, 1/q\right)$

Proof

$$\begin{tikzcd} L_n \arrow[r, "\Theta^{-1}"] & \mathbb{Z}^n_{\geq 0} \arrow[d, "\text{reverse}"', "{\begin{matrix} p_1 p_2 \cdots p_n \\ \downarrow \\ p_n p_{n-1} \cdots p_1 \end{matrix}}"] & \\ & \mathbb{Z}^n_{\geq 0} \arrow[r, "\Phi"] & A_n \end{tikzcd}$$

Details :

$$\overset{\lambda}{\overbrace{i\,w_i - \epsilon_i(\pi)}} \longleftarrow w_1 \le \dots \le w_n \xleftarrow{\pi^{-1}} p_1 p_2 \dots p_n \qquad : \ominus$$

$$\Phi : \quad p_n p_{n-1} \cdots p_1 \xrightarrow{\sigma^{-1}} w_n \geq \dots \geq w_1 \longrightarrow \overset{\lambda'}{\overbrace{(n+1-i)\,w_i + \epsilon_i(\sigma)}}$$

$$u^{|w|} q^{|\lambda|} = u^{|w|} q^{\sum i w_i - \mathrm{inv}(\pi)} \longleftarrow \boxed{q = 1/q \, ; \quad u = u q^{n+1}}$$

$$u^{|w|} q^{|\lambda'|} = u^{|w|} q^{\sum (n+1-i) w_i + \mathrm{inv}(\sigma)}$$

Permutation Statistics

$\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in S_n$

A descent of π is a position i s.t. $\pi_i > \pi_{i+1}$

$D(\pi) = \{ i \mid \pi_i > \pi_{i+1} \}$ descent set

$$\operatorname{des}(\pi) = \sum_{i \in D(\pi)} 1$$

$$\operatorname{maj}(\pi) = \sum_{i \in D(\pi)} i$$

Quadratic Permutation Statistics

$\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in S_n$

A descent of π is a position i s.t. $\pi_i > \pi_{i+1}$

$D(\pi) = \{ i \mid \pi_i > \pi_{i+1} \}$ descent set

$\mathrm{des}(\pi) = \sum_{i \in D(\pi)} 1$

$\mathrm{sq}(\pi) = \sum_{i \in D(\pi)} i^2$

$\mathrm{maj}(\pi) = \sum_{i \in D(\pi)} i$

$\mathrm{bin}(\pi) = \sum_{i \in D(\pi)} \binom{i+1}{2}$

Theorem [MacMahon]

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = \prod_{i=1}^{n} [i]_q$$

where $[i]_q$ is the q-analog of integer i :

$$[i]_q = \frac{1-q^i}{1-q} = 1 + q + q^2 + \dots + q^{i-1}$$

The proof of MacMahon's theorem involves counting the points in $\mathbb{Z}_{\geq 0}^n$ in two ways:

(1) $$\sum_{p \in \mathbb{Z}_{\geq 0}^n} q^{|p|} = \frac{1}{(1-q)^n}$$

(2) Partition $\mathbb{Z}_{\geq 0}^n$ by associating points $p \in \mathbb{Z}_{\geq 0}^n$ with the permutation π that sorts them:

$$\sum_{\pi \in S_n} \sum_{\substack{p \text{ s.t.} \\ \pi(p) \text{ is sorted} \\ \text{and } \dots\dots}} q^{|p|} = \frac{\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)}}{\prod_{i=1}^{n} (1-q^i)}$$

Equate (1) and (2).

Try to imitate - count A_n in two ways:

(1) $$\sum_{\lambda \in A_n} u^{||\lfloor \lambda_n \rfloor||} q^{|\lambda|} = \prod_{i=1}^{n} \frac{1+uq^i}{1-u^2 q^{i+1}}$$ (refined ALHCT)

(2) Use Φ. Partition the points in $\mathbb{Z}^n_{\geq 0}$ according to the permutation that reverse-sorts them. Then map them to L_n via Φ.

$$\sum_{\pi \in S_n} \sum_{\substack{p \text{ s.t.} \\ \pi(p) \text{ is sorted}}} u^{|p|} q^{|\Phi(p)|} = \cdots \frac{\sum_{\pi \in S_n} \boxed{}}{\prod_{i=1}^{n}}$$

Equate (1) and (2)

Step (2):

$$A_n(u, q) = \frac{\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} \prod_{i \in D(\pi)} u^i q^{i(i+1)/2}}{\prod_{i=1}^{n} (1 - u^i q^{i(i+1)/2})}$$

Step (2):

$$A_n(u,q) = \frac{\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} \prod_{i \in D(\pi)} u^i q^{i(i+1)/2}}{\prod_{i=1}^{n} \left(1 - u^i q^{i(i+1)/2}\right)}$$

$$= \frac{\sum_{\pi \in S_n} u^{\mathrm{maj}(\pi)} q^{\mathrm{inv}(\pi) + \mathrm{bin}(\pi)}}{\prod_{i=1}^{n} \left(1 - u^i q^{i(i+1)/2}\right)}$$

Equate with ALHCT

Theorem A [Bright, S 2009]

$$\sum_{\pi \in S_n} u^{\mathrm{maj}(\pi)} q^{\mathrm{inv}(\pi) + \mathrm{bin}(\pi)} = \prod_{i=1}^{n} \frac{1+uq^i}{1-u^2q^{i+1}} \left(1 - u^i q^{i(i+1)/2}\right)$$

Theorem A [Bright, S 2009]

$$\sum_{\pi \in S_n} u^{\mathrm{maj}(\pi)} q^{\mathrm{inv}(\pi) + \mathrm{bin}(\pi)} = \prod_{i=1}^{n} \frac{1+uq^i}{1-u^2q^{i+1}} \left(1 - u^i q^{i(i+1)/2}\right)$$

Corollary. Set $q=1$

$$\sum_{\pi \in S_n} u^{\mathrm{maj}(\pi)} = \prod_{i=1}^{n} [i]_u$$

Theorem A [Bright, S 2009]

$$\sum_{\pi \in S_n} u^{\mathrm{maj}(\pi)} q^{\mathrm{inv}(\pi) + \mathrm{bin}(\pi)} = \prod_{i=1}^{n} \frac{1+uq^i}{1-u^2 q^{i+1}} \left(1 - u^i q^{i(i+1)/2}\right)$$

Corollary [A1] set $u=1$

$$\sum_{\pi \in S_n} q^{\mathrm{bin}(\pi) + \mathrm{inv}(\pi)} = \prod_{i=1}^{n} [2]_{q^i} \frac{[i(i+1)/2]_q}{[i+1]_q}$$

Theorem A [Bright, S 2009]

$$\sum_{\pi \in S_n} u^{\mathrm{maj}(\pi)} q^{\mathrm{inv}(\pi) + \mathrm{bin}(\pi)} = \prod_{i=1}^{n} \frac{1+uq^i}{1-u^2q^{i+1}} \left(1-u^i q^{i(i+1)/2}\right)$$

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Theorem A [Bright, S 2009]

$$\sum_{\pi \in S_n} u^{\mathrm{maj}(\pi)} q^{\mathrm{inv}(\pi)+\mathrm{bin}(\pi)} = \prod_{i=1}^{n} \frac{1+uq^i}{1-u^2q^{i+1}} \left(1-u^i q^{i(i+1)/2}\right)$$

Corollary [A2] set $q = q^{-1}$ and $u = q^{n+1}$

$$\sum_{\pi \in S_n} q^{(n+1)\mathrm{maj}(\pi) - \mathrm{bin}(\pi) - \mathrm{inv}(\pi)} = \prod_{i=1}^{n} [i]_{q^{2(n-i)+1}}$$

Other results about the distribution of quadratic perm. stats.?

Theorem B [Bright, S 2009]

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{sq}(\pi)+\mathrm{inv}(\pi)} = \prod_{i=1}^{n} [i]_{qt^i}$$

Proof - Elementary, adapted from [Zabrocki 2003]

Theorem B [Bright, S 2009]

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{sq}(\pi) + \mathrm{inv}(\pi)} = \prod_{i=1}^{n} [i]_{qt^i}$$

Corollary B.1 $q=1$

$$\sum_{\pi \in S_n} t^{\mathrm{sq}(\pi) + \mathrm{inv}(\pi)} = \prod_{i=1}^{n} [i]_{t^i}$$

Theorem B [Bright, S 2009]

$$\sum_{\pi \in S_n} q^{maj(\pi)} t^{sq(\pi)+inv(\pi)} = \prod_{i=1}^{n} [i]_{qt^i}$$

Corollary B.2 $q = q^n$ $t = 1/q$ [Zabrocki 03] from [Stembridge + Waugh 98]

$$\sum_{\pi \in S_n} q^{n \cdot maj(\pi) - sq(\pi) - inv(\pi)} = \prod_{i=1}^{n} [i]_{q^{n-i}}$$

Theorem B [Bright, S 2009]

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{sq}(\pi) + \mathrm{inv}(\pi)} = \prod_{i=1}^{n} [i]_{qt^i}$$

Corollary B.3 $q = q^{2n+1}$ $t = 1/q^2$

$$\sum_{\pi \in S_n} q^{(2n+1)\,\mathrm{maj}(\pi) - 2\,\mathrm{sq}(\pi) - 2\,\mathrm{inv}(\pi)} = \prod_{i=1}^{n} [i]_{q^{2(n-i)+1}}$$

Notice how sq / inv and bin / inv

always appear together

Propose :

$$\mathrm{sqin}(\pi) = \mathrm{sq}(\pi) + \mathrm{inv}(\pi)$$

$$\mathrm{binv}(\pi) = \mathrm{bin}(\pi) + \mathrm{inv}(\pi)$$

Overview

(A.1) $$\sum_{\pi \in S_n} q^{\mathrm{binv}(\pi)} = \prod_{i=1}^{n} \frac{[2]_{q^i} \left[\binom{i+1}{2}\right]_q}{[i+1]_q}$$

(B.1) $$\sum_{\pi \in S_n} q^{\mathrm{sqin}(\pi)} = \prod_{i=1}^{n} [i]_{q^i}$$

Overview

(A.1) $$\sum_{\pi \in S_n} q^{\mathrm{binv}(\pi)} = \prod_{i=1}^{n} \frac{[2]_{q^i} \left[\binom{i+1}{2}\right]_q}{[i+1]_q}$$

(A.2) $$\sum_{\pi \in S_n} q^{(n+1)\mathrm{maj}(\pi) - \mathrm{binv}(\pi)} = \prod_{i=1}^{n} [i]_{q^{2(n-i)+1}}$$

(B.1) $$\sum_{\pi \in S_n} q^{\mathrm{sqin}(\pi)} = \prod_{i=1}^{n} [i]_{q^i}$$

(B.2) $$\sum_{\pi \in S_n} q^{n\,\mathrm{maj}(\pi) - \mathrm{sqin}(\pi)} = \prod_{i=1}^{n} [i]_{q^{n-i}}$$

A few more observations

(A.2)
$$\sum_{\pi \in S_n} q^{(n+1)\mathrm{maj}(\pi) - \mathrm{binv}(\pi)} = \prod_{i=1}^{n} [i]_{q^{2(n-i)+1}}$$

(B.3)
$$\sum_{\pi \in S_n} q^{(2n+1)\mathrm{maj}(\pi) - 2\,\mathrm{sgin}(\pi)} = \prod_{i=1}^{n} [i]_{q^{2(n-i)+1}}$$

(L)
$$\sum_{\pi \in S_n} q^{\binom{n+1}{2}\mathrm{des}(\pi) - \mathrm{binv}(\pi)} = \prod_{i=1}^{n} [i]_{q^{2(n-i)+1}}$$

Theorem [Stembridge & Waugh 1998] (*)

" Let W be a finite Weyl group ...

⋮

... By analyzing the structure of the corresponding affine Weyl group we prove:

$$\sum_{w \in W} q^{\sigma(w) - \ell(w)} = f \prod_{i=1}^{n} \frac{1-q^{b_i}}{1-q^{e_i}}$$ "

" A Weyl group generating function that ought to be better known "

Further Directions

I. Connections with Weyl / Coxeter groups

(a) Interpret QPS distributions as statements about affine Coxeter groups (see (*))

(b) LPT arose in this theory. Where do ALC fit in? The geometry of anti-lecture hall compositions seems more natural, but it gives rise to a strange symmetric group generating function

(c) Combinatorial generalizations and refinements of LHP thms $\xrightarrow{\text{QPS}}$ interesting observations about affine Coxeter gps?

Further Directions

II. Quadratic Permutation Statistics

(a) Does Theorem A have an elementary proof like Theorem B?

(b) Equidistribution results (e.g. $\sum q^{\mathrm{maj}(\pi)} = \sum q^{\mathrm{inv}(\pi)}$)

for quadratic permutation statistics? (Calculations say yes, some)

(c) Use known generalizations and refinements of lecture hall theorems to refine QPS distributions

Further Directions

III. Lecture hall partitions and anti-lecture hall compositions.

(a) There is a third parameter in the refined LHPT & ALHCT that one might track:

$$L_n(v,u,q) = \prod_{i=1}^{n} \frac{1+uvq^i}{1-u^2q^{n+i}}$$

v tracks # of odd parts of $\lceil \lambda \rceil$

(b) Can the "geometry" give new insight into Euler's partition theorem?

(c) Or into other q-series identities related to LHP & ALHC:
q-Chu-Vandermonde, q-Gauss summation, little Göllnitz theorems.

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