Compositions and Infinite Matrices

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9 Feb 2013

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Definition

Let $n \geq 0$. A *composition* of the integer n is an ordered tuple of positive integers $(a_1, \ldots, a_k)$ whose sum is n:

$$n = \sum_{i=1}^{k} a_i$$

Theorem 1

For $n, k \geq 1$ the number of compositions of n into k parts is

$$\binom{n-1}{k-1}$$

Proof: Composition $(a_1, \ldots, a_k)$ corresponds to subset

$$1 \leq a_1 < a_1 + a_2 < \cdots < a_1 + \cdots + a_{k-1} \leq n-1.$$

Unrestricted Compositions are too Simple

Carlitz (1976): adjacent parts must be unequal.

How many of these are there ?

$$\begin{array}{ccc} 7\,(1) & 6+1\,(2) & 5+2\,(2) \\ 5+1+1\,(1) & 4+3\,(2) & 4+2+1\,(6) \\ 3+3+1\,(1) & 3+2+2\,(1) & 3+2+1+1\,(6) \\ 2+2+1+1+1\,(1) & & \end{array}$$

A003242 Number of compositions of n such that no two adjacent parts are equal. $1, 1, 1, 3, 4, 7, 14, 23, 39, 71, 124, 214,$ $378, 661, 1152, 2024, 3542, 6189, 10843$

Exponential Growth

How many walks are there of length n on $\mathbb{Z}^2$?
Answer: 4^n.

How many of these, say SAW_n, are self avoiding ?

Proposition. The limit

$$\lim_{n \to \infty} (\mathrm{SAW}_n)^{1/n}$$

exists.

Proof. Use Fekete's Lemma.

Fekete's Lemma

Suppose that c_n is a sequence of positive numbers satisfying

$$c_{m+n} \leq c_m c_n.$$

Then the limit

$$\lim_{n \to \infty} (c_n)^{1/n}$$

exists.

What happens when we try to apply this to

$$c_n := \#\,\text{Carlitz compositions}.$$

Theorem 2 (Knopfmacher & Prodinger, 1998)

For suitable constants A and $1/2 < r < 1$, the number c_n of Carlitz compositions satisfies

$$c_n = Ar^{-n}(1 + o(1)).$$

- $(c_n)^{1/n} \to \frac{1}{r}$
- $o(1)$ goes to zero exponentially fast
- r is the unique root in the interval $(0, 1)$ of the equation

$$\frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^3}{1-x^3} - \cdots \quad = \quad 1.$$

Power Series Remarks

$$\sum_{n=0}^{\infty} c_n z^n$$

has a radius of convergence r

$r = \sup\{\rho : c_n \rho^n \to 0\}$

has a singularity on the circle of convergence
[What *is* a singularity ?]

$c_n \geq 0$ implies $z = r$ is a singularity (Pringsheim's Theorem)

Theorem 3

Let $c_n \geq 0$ be a sequence, and assume that the power-series

$$\sum_{n=0}^{\infty} c_n z^n$$

has radius of convergence r, and no singularity on its circle of convergence other than a simple pole at $z = r$. Then, for suitable constant A,

$$c_n = Ar^{-n}(1 + o(1))$$

with the $o(1)$ term exponentially small.

What is a simple pole ?

Let r be the radius of convergence of the power-series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

We say $f(z)$ *has a simple pole at* $z = r$ provided

$$f(z) = \frac{A}{1 - z/r} + g(z)$$

for all z satisfying

$$z \in \{|z| < r\} \cap \{|z - r| < \delta\}$$

with $g(z)$ analytic in $\{|z - r| < \delta\}$.

Proof of Theorem 3

The hypotheses imply that the difference

$$f(z) - \frac{A}{1 - z/r}$$

is analytic in an open disc containing a circle $|z| = r + \delta$, $\delta > 0$. Consequently, by Cauchy's integral formula

$$\left| [z^n] \left(f(z) - \frac{A}{1 - z/r} \right) \right| \leq K \times (r + \delta)^{-n}$$

You may take the constant K as

$$K = \max_{|z| = r + \delta} \left| f(z) - \frac{A}{1 - z/r} \right| .$$

What is Cauchy's Integral Formula ?

$$[z^n]f(z) = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z^{n+1}} dz$$

provided $f(z)$ is a power series whose radius of convergence exceeds R.

Proof of Theorem 2

Theorem 2, due to K&P, '98, is a consequence of

- Theorem 3
- Carlitz's 1976 generating function

$$\sum_{n=0}^{\infty} c_n x^n = \frac{1}{1-\sigma(x)},$$

where

$$\sigma(x) = \frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^3}{1-x^3} - \cdots .$$

- and some numerics.

Other Restrictions

Idea: Think of other classes of compositions to investigate.

It seemed like the Carlitz compositions were in analogy with partitions; instead of requiring

$$a_1 + \cdots + a_k = n, \quad a_1 \geq a_2 \geq \cdots \geq a_k$$

the Carlitz compositions require

$$a_1 + \cdots + a_k = n, \quad a_1 \neq a_2 \neq \cdots \neq a_k$$

Two Rows

Macmahon looked at two-rowed partitions

$$\begin{array}{cccc} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{array}$$

$$\sum_i (a_i + b_i) = n, \quad a_i \geq a_{i+1}, \quad b_i \geq b_{i+1}, \quad a_i \geq b_i$$

and found the ordinary generating function

$$(1-x)^{-1} \prod_{i=2}^{\infty} (1-x^i)^{-2} = 1+x+3x^2+5x^3+10x^4+16x^5+29x^6+\cdots$$

Two Rowed Compositions

$$\begin{array}{cccc} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{array}$$

$$\sum_i (a_i + b_i) = n, \quad a_i \neq a_{i+1}, \quad b_i \neq b_{i+1}, \quad a_i \neq b_i$$

the ordinary generating function (disallowing zero as a part)

$$2x^3 + 2x^4 + 4x^5 + 6x^6 + 10x^7 + \cdots$$

The Missing Formula

	Partitions	*Analogous Compositions*
One row	$\prod_{i=1}^{\infty}(1-x^i)^{-1}$	$\left(1+\sum_{i=1}^{\infty}\frac{(-1)^i x^i}{1-x^i}\right)^{-1}$
Two rows	$(1-x)^{-1}\prod_{i=2}^{\infty}(1-x^i)^{-1}$	???

A Simpler(?), at least Different, Generalization

No part a_i of the composition can be equal to either of its neighbors, or to either of its one-away neighbors.

$$\text{Bad}: \quad 7+3+3+4, \quad 7+3+7+4$$
$$\text{Good}: \quad 7+3+4+7$$

We might call these
Distance-2 Carlitz compositions.

$$x + x^2 + 3x^3 + 3x^4 + 5x^5 + 11x^6 + \cdots$$

Graphlike Restrictions

The parts a_i are placed on the vertices of a graph; no two parts joined by an edge can be equal.

Unrestricted: edgeless graph
Carlitz: a path
Distance-2 Carlitz: path $+$ chords $=$ triangular ladder
Two-rowed: ladder

Another Sort of Restriction

A composition $(a_1, a_2, \ldots, a_k)$ is ALTERNATING if either

$$a_1 > a_2 < a_3 > \cdots$$

or

$$a_1 < a_2 > a_3 < \cdots$$

The Moving Window Criterion

Study restrictions on compositions that are LOCAL

The class $\mathcal{C}$ of compositions is a LOCALLY RESTRICTED class if there is a window size m such that membership in $\mathcal{C}$ can be tested by sliding a window across the composition and finding that it passes a local test at each position.

Technicalities:
(1) the test-function can depend on window-position mod m'
(2) assuming implicit leading zeros, special starting conditions can be imposed
(3) likewise finishing

Example: Two-rowed Compositions

The window presumes a linear arrangement, so write

$$\begin{matrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{matrix}$$

as

$$a_1 \quad b_1 \quad a_2 \quad b_2 \quad \cdots \quad a_k \quad b_k$$

Window size is 3
Window $= [a, b, c]$ and $c \neq 0 \implies c \neq a$
If, in addition, window position (c) is even, $c \neq b$
$[a, b, c]$, $c = 0, b \neq 0 \implies a \neq 0$, position is odd

Building Locally Restricted Compositions

Local restrictions can be incorporated by building the compositions out of sufficiently long segments.

Call the "sufficiently long" quantity m, the word-size.

A *word* is a composition of length m.

You need a binary relation which tells you when two words $\mathbf{a}$, $\mathbf{b}$ can appear next to each other in a composition.

Issue: maybe not all compositions have lengths a multiple of m.

The Digraph Criterion

We have a digraph $D = (V, E)$
The vertex set V is a set of words
Some vertices are legal starting vertices
Some vertices are legal finishing vertices
Legal non-empty compositions are in 1-to-1 correspondance with walks:

$$\mathbf{b}_1 \to \mathbf{b}_2 \to \cdots \to \mathbf{b}_\ell \quad \text{(walk)}$$

corresponds to

$$\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_\ell \quad \text{(composition)}$$

Implications

With proper formulation,

$$\text{Graphlike} \Longrightarrow \text{Moving window} \Longrightarrow \text{Digraph}$$

An Infinite Matrix

Rows and columns are indexed by nonzero words $\mathbf{a}, \mathbf{b}, \ldots$

$$T_{\mathbf{a},\mathbf{b}} = \begin{cases} x^{\Sigma(\mathbf{b})} & \text{if } \mathbf{b} \text{ can follow } \mathbf{a} \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{n=1}^{\infty} c_n x^n = \vec{\mathbf{s}}(x)\,(I + T + T^2 + \cdots)\,\vec{\mathbf{f}}(x)$$

where $\vec{\mathbf{s}}(x)$ is an infinite row, $\vec{\mathbf{f}}(x)$ an infinite column.

The Matrix Sum

$$\sum_{n=1}^{\infty} c_n x^n = \vec{\mathbf{s}}(x)\,(I + T + T^2 + \cdots)\,\vec{\mathbf{f}}(x)$$

should be the formal sum over all legal word concatenations

$$\mathbf{b}_1\,\mathbf{b}_2\,\cdots\,\mathbf{b}_\ell \qquad (\ell \geq 1)$$

of the weight

$$x^{\Sigma(\mathbf{b}_1)+\cdots+\Sigma(\mathbf{b}_\ell)}.$$

Thus,

$$\vec{\mathbf{s}}_{\mathbf{a}}(x) = \begin{cases} x^{\Sigma(\mathbf{a})} & \text{if } \mathbf{a} \text{ can start a composition} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\vec{\mathbf{f}}_{\mathbf{b}}(x) = \begin{cases} 1 & \text{if } \mathbf{b} \text{ can end a composition} \\ 0 & \text{otherwise} \end{cases}$$

Making Formal Sense of the Matrix Sum

$$\sum_{n=1}^{\infty} c_n x^n = \vec{\mathbf{s}}(x)\,(I + T + T^2 + \cdots)\,\vec{\mathbf{f}}(x) \qquad (*)$$

Start with an infinite sequence of weights

$$x^{\Sigma(\mathbf{a})}, x^{\Sigma(\mathbf{b})}, \ldots$$

and use these to create $\vec{\mathbf{s}}(x)$, $T, \vec{\mathbf{f}}(x)$.

The row $\vec{\mathbf{s}}(x)$ is some subset of the given sequence, missing elements replaced by zeros;
$T_{\mathbf{a},\mathbf{b}}$ is $x^{\Sigma(\mathbf{b})}$ for some pairs $(\mathbf{a}, \mathbf{b})$ and zero for others;
and $\vec{\mathbf{f}}(x)$ is an arbitrary column vector of zeros and ones.

Does the matrix equation $(*)$ yield a well defined sequence c_n ?

Making Formal Sense of the Matrix Sum, continued

Proposition: If for each n the set

$$\{\mathbf{a} : \Sigma(\mathbf{a}) \leq n\}$$

is finite, then the matrix equation

$$\sum_{n=1}^{\infty} c_n x^n = \vec{\mathbf{s}}(x)\,(I + T + T^2 + \cdots)\,\vec{\mathbf{f}}(x)$$

makes sense as a formal generating function.

Reminder Here, the three objects $\vec{\mathbf{s}}(x), T, \vec{\mathbf{f}}(x)$ are assumed to be created from the infinite sequence of weights

$$x^{\Sigma(\mathbf{a})}, x^{\Sigma(\mathbf{b})}, \ldots$$

as described on the previous slide.

Some Modest Questions

$$\sum_{n=1}^{\infty} c_n x^n = \vec{\mathbf{s}}(x)\,(I + T + T^2 + \cdots)\,\vec{\mathbf{f}}(x)$$

When $\vec{\mathbf{s}}(x)$, $T, \vec{\mathbf{f}}(x)$ arise from LR compositions,

$$c_n \leq 2^{n-1}$$

for all n. In cases of interest,

$$c_n \geq 1$$

for infinitely many n. And so, the power-series on the left has a radius of convergence r satisfying

$$\frac{1}{2} \leq r \leq 1$$

How is r determined from the equation ? When might $x = r$ be the only singularity on the circle of convergence ? When might $x = r$ be a simple pole ?

Recurrent Words

Our compositions are concatenations of words

$$\mathbf{a} = (a_1, a_2, \ldots, a_m)$$

$a_1 \neq 0$; 0's, if any, at the end.

(Need only consider those words which occur.)

A word $\mathbf{a}$ is RECURRENT if it can appear in the same composition twice.

Recurrent words appear arbitrarily often.

Theorem 4

Hypotheses.
Given: a class $\mathcal{C}$ of locally restricted compositions determined by a digraph $D = (V, E)$ whose vertices are words of length m.

1. There are at least two recurrent words.

2. The recurrent words form a stronly connected digraph among themselves.

3. No composition contains more than K nonrecurrent words.

Another Hypothesis

4. For every pair of recurrent words **b**, **c** there exists a length $\ell \geq 1$ and a recurrent word **a** s.t.

$$T^{\ell}_{\mathbf{ab}} \neq 0, \quad T^{\ell}_{\mathbf{ac}} \neq 0.$$

The Final Hypothesis

5. There is an integer $k > 0$ and (possibly equal) recurrent vertices $\mathbf{a}$ and $\mathbf{b}$ such that when S is defined to be the set of sums

$$\Sigma(\mathbf{a}) + \Sigma(\mathbf{c}_1) + \cdots + \Sigma(\mathbf{c}_{k-1}) + \Sigma(\mathbf{b})$$

over all k-edged subcompositions connecting $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{a} \quad \mathbf{c}_1 \quad \cdots \quad \mathbf{c}_{k-1} \quad \mathbf{b}$$

we have

$$\gcd\{m - n : m \in S, n \in S\} = 1.$$

Altogether . . .

Let $\mathcal{C}$ be a collection of locally restricted compositions determined by a (possibly infinite) digraph on words, and whose recurrent words satisfy the five stated hypotheses.

Then, the ordinary generating function

$$\sum_{n=1}^{\infty} c_n x^n$$

- has a simple pole at $x = r$
- has no other singularity on the circle of convergence.

What is the radius of convergence r ?

Let $\nu_1, \nu_2, \ldots$ be a listing of the recurrent words, and now let (the new) transfer matrix be based only on these:

$$T_{\nu,\nu'} = \begin{cases} x^{\Sigma(\nu)+\Sigma(\nu')} & \text{if } \nu' \text{ can follow } \nu \\ 0 & \text{otherwise} \end{cases}$$

$$r = \sqrt{x_0}$$

where x_0 is determined by

$$\text{spectral radius}\,(T(x_0)) = 1.$$

Questions!

What is the spectral radius of a (possibly) infinite matrix ?

What happened to the straightforward transfer matrix T that I could understand ?

The Spectral Radius

The limit

$$\lim_{n\to\infty} \|T^n\|^{1/n}$$

exists, and is called the *spectral radius* of T.

This is proven à la Fekete using

$$\|T^{m+n}\| \le \|T^m\| \, \|T^n\|.$$

Operator norm,

$$\|T\| \stackrel{\text{def}}{=} \sup_{\|\mathbf{v}\|=1} \|T\mathbf{v}\|.$$

Please, tell me more about the spectral radius

For our infinite matrix $T(x)$, let $0 < x_0 < 1$ solve

$$\text{spectral radius}(T(x_0)) = 1.$$

Then, as in standard complex variables,

$$I + T(x) + T(x)^2 + \cdots$$

is analytic for $|x| < x_0$ and has a singularity on $|x| = x_0$. Moreover, since $T(x) \geq 0$, Pringsheim's theorem applies, and $x = x_0$ is a singularity.

Count 'Em Coming and Going

Even with the old T, with a slight change

$$T_{\mathbf{a},\mathbf{b}} = \begin{cases} x^{\Sigma(\mathbf{a})+\Sigma(\mathbf{b})} & \text{if } \mathbf{b} \text{ can follow } \mathbf{a} \\ 0 & \text{otherwise} \end{cases}$$

we have

$$\sum_{n=1}^{\infty} c_n x^{2n} = \vec{\mathbf{s}}(x) \left(I + T + T^2 + \cdots \right) \vec{\mathbf{f}}(x)$$

($\vec{\mathbf{f}}(x)$ is a little different, too.)

But why, why would you do that ?

The Reason Why

$$\sum_{n=1}^{\infty} c_n x^{2n} = \vec{\mathbf{s}}(x)\,(I + T + T^2 + \cdots)\,\vec{\mathbf{f}}(x)$$

With the modified definition of the transfer matrix

we lose $F(x)$, but we have $F(x^2)$
and now the entries in the matrix T are (square) summable

This implies that T is a compact operator on Hilbert Space ℓ^2.

The importance of being compact $\cdots$

Compact Operators

Nonzero spectral values are isolated and are eigenvalues

Are the closure in the Banach space of bounded linear operators of the set of finite rank operators

Compactness + nonnegative $\Longrightarrow$ Krein-Rutman is applicable

BTW, Definition: A compact if

$$\mathbf{v}_k \text{bounded} \quad \Longrightarrow \quad A\mathbf{v}_k \text{ contains a convergent subsequence.}$$

Getting the Simple Pole

For each x, $0 < x < 1$, KR+hypotheses imply $T(x)$ has a dominant eigenvalue equal to its spectral radius whose eigenvector is simple and strictly positive.

$$T = \lambda E + B$$

$$E = E^2, \quad \mathrm{rank}(E) = 1$$

(E is projection onto a one-dimensional subspace)

$$EB = BE = 0$$

$$\mathrm{spr}(B) < \mathrm{spr}(T)$$

There is an interesting formula for the projection E.

The Formula for Projection E

Let Γ be a small circle enclosing nonzero eigenvalue λ for compact operator T. Then,

$$E = \frac{1}{2\pi i} \oint_{\Gamma} \frac{dz}{zI - T} \qquad (*)$$

is a projection which commutes with T and whose image is the finite dimensional eigenspace associated with λ.

Tosio Kato, author of *Perturbation Theory for Linear Operators*, says in a footnote on page 67 that the integral formula $(*)$ is "basic throughout the present book."

This provides the key to . . .

A Lemma

For each x_0, $0 < x_0 < 1$, there is a neighborhood $\mathcal{N}$ of x_0 such that for $x \in \mathcal{N}$

$$T(x) = \lambda(x)E(x) + B(x)$$

$$E(x) = E(x)^2, \quad \mathrm{rank}(E(x)) = 1$$

($E(x)$ is projection onto a one-dimensional subspace)

$$E(x)B(x) = B(x)E(x) = 0$$

$$\mathrm{spr}(B(x)) < \mathrm{spr}(T(x))$$

Note:

$$T(x)^k = \lambda(x)^k E(x) + B(x)^k$$

And so,

In the neighborhood $\mathcal{N}$

$$
\begin{aligned}
& \vec{\mathbf{s}}(x)\left(I + T(x) + T(x)^2 + \cdots\right)\vec{\mathbf{f}}(x) \\
= \quad & \vec{\mathbf{s}}(x)\left(\frac{\lambda(x)}{1-\lambda(x)}E(x) + (I - B(x))^{-1}\right)\vec{\mathbf{f}}(x) \\
= \quad & \frac{\lambda(x)}{1-\lambda(x)}\vec{\mathbf{s}}(x)E(x)\vec{\mathbf{f}}(x) \ + \ \vec{\mathbf{s}}(x)(I - B(x))^{-1}\vec{\mathbf{f}}(x) \\
= \quad & \frac{g(x)}{1-\lambda(x)} + h(x).
\end{aligned}
$$

What Was That Hypothesis 5 All About ?

$$\gcd(n_2 - n_1, \ldots, n_\ell - n_1) = 1, \quad a_i > 0, \quad x \notin [0, \infty)$$

implies

$$\left| \sum_{i=1}^{\ell} a_i x^{n_i} \right| \quad < \quad \sum_{i=1}^{\ell} a_i |x|^{n_i}.$$

The gcd-condition (Hypothesis 5) gives k,**a**,**b** s.t.

$$\left| [T(x)^k]_{\mathbf{a},\mathbf{b}} \right| \quad < \quad \left| [T(|x|]^k)_{\mathbf{a},\mathbf{b}} \right|$$

for $x \notin [0, \infty)$; whence, no other singularities.

Papers

Restricted Adjacent Differences	2005	27 *pp*
General Restrictions and Infinite Matrices	2009	36 *pp*
Adjacent – Part Periodic Inequalities	2010	9 *pp*
Nearly Free Large Parts and Gap – Freeness	2012	29 *pp*

simple pole on its circle of convergence
number of compositions asymptotic to Ar^{-n}
multivariate central and local limit theorems, involving number of parts, numbers of parts of given sizes, and number of rises and fall, ...
the largest part
the number of distinct part
the length of the longest run
Poisson distribution of large parts, gap freeness

Now What

Is all that published stuff right
Can anything be done with the simple (first) matrix equation
How about GF's for Distance-2 and/or Two-rowed
Can GF's arising from infinite matrices have other singularities
Can you use infinite matrices to work a hard problem
Get from the infinite matrix approach to Carlitz compositions to the equation satisfied by the radius of convergence