Triangle Lectures in Combinatorics

NC State University

Raleigh, NC

f-vectors, descents sets and the weak order

E. Swartz
Cornell University, Ithaca, NY.
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What do these complexes have in common?

- Order complexes
  - Distributive lattices
  - Geometric lattices
  - Face posets of CM complexes
- Finite buildings

Enumeration

$f_i = \#$ of i-dimensional faces.

For order complexes this equals $\#$ of chains

$$x_0 < \cdots < x_i$$

of length i.

$(f_0, f_1, \ldots, f_{d-1})$ is the f-vector.

All of these complexes are completely balanced. They can be colored with d colors.

The flag f-vector is the set of all $f_A,\ A \subseteq [d]$.

$$f_A = \# \text{ faces whose colors equal } A.$$

Notice

$$f_i = \sum_{|A|=i+1} f_A.$$

<u>h-vectors</u>

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}.$$

Ex: If Δ is four dimensional, then $d = 5$ and

$$h_3 = f_2 - 3f_1 + 6f_0 - 10.$$

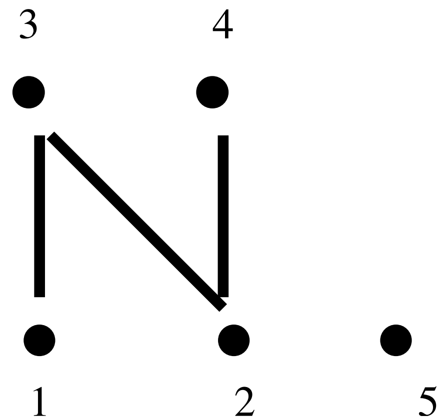
Flag h-vectors are defined by

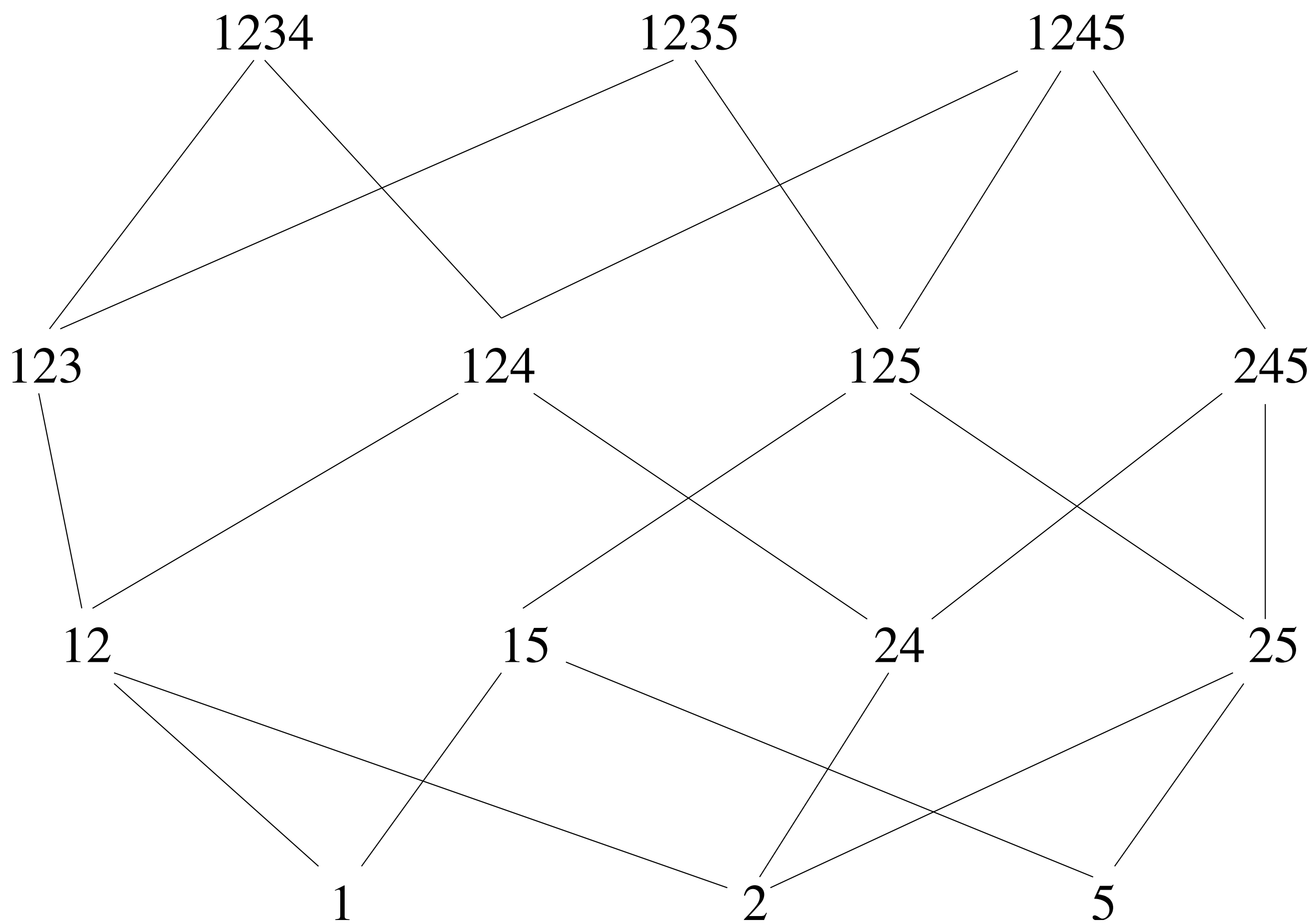
$$h_A = \sum_{B \subseteq A} (-1)^{|A-B|} f_B.$$

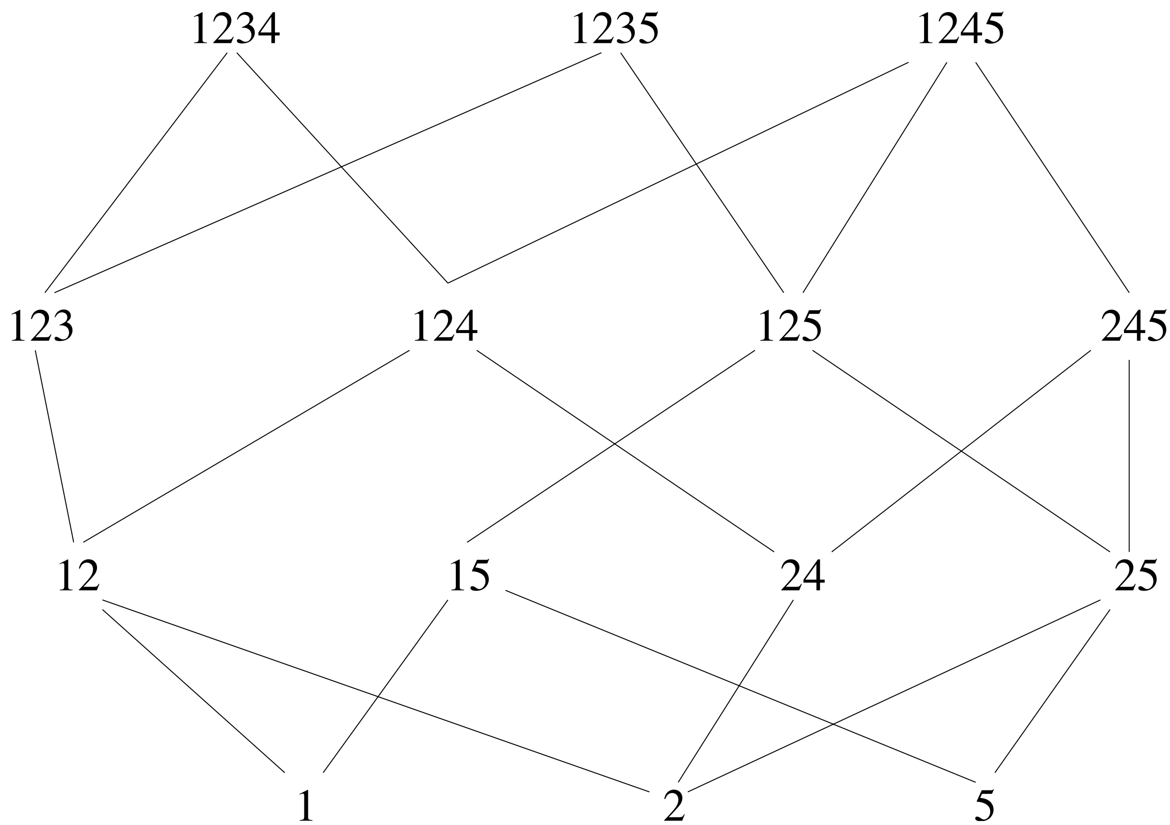
Inclusion-exclusion implies

$$\sum_{|A|=i} h_A = h_i.$$

Example: A distributive lattice







$f_\emptyset = 1$	$h_\emptyset = 1$		
$f_{\{1\}} = 3$	$f_{\{1,2\}} = 7$	$h_{\{1\}} = 2$	$h_{\{1,2\}} = 1$
$f_{\{2\}} = 4$	$f_{\{1,3\}} = 9$	$h_{\{2\}} = 3$	$h_{\{1,3\}} = 3$
$f_{\{3\}} = 4$	$f_{\{2,3\}} = 8$	$h_{\{3\}} = 3$	$h_{\{2,3\}} = 1$
$f_{\{4\}} = 3$	$f_{\{1,4\}} = 8$	$h_{\{4\}} = 2$	$h_{\{1,4\}} = 3$
	$f_{\{2,4\}} = 9$		$h_{\{2,4\}} = 3$
	$f_{\{3,4\}} = 7$		$h_{\{3,4\}} = 1$

Theorem 1 *If Δ is a finite building, order complex of a geometric lattice or the order complex of a rank selected face poset of a CM complex which does not contain the top rank, then*

$$h_i \leq h_{d-i},\ i \leq d/2.$$

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}.$$

Theorem 2 *If Δ is the order complex of a distributive lattice, then*

$$h_i \geq h_{d-i},\ i \leq d/2.$$

$$h_d \leq h_{d-1} \leq \cdots \leq h_{d-\lfloor d/2 \rfloor}.$$

The proofs of these theorems all depend on commutative algebra and the g-theorem for Coxeter complexes.

The first theorem also uses Chari's convex ear decompositions. This is a method for decomposing a complex into understandable pieces. In each case the complex can be decomposed into subcomplexes each of which is a shellable ball which is itself a subcomplex of a finite Coxeter complex.

The order complex of a distributive lattice is itself a shellable ball which is a subcomplex of a finite Coxeter complex.

The weak order

(W, S) a finite Coxeter system.

W is a finite group with generators $S = \{s_1, \ldots, s_n\}$ and relations $s_i^2 = 1, (s_i s_j)^{m_{i,j}} = 1$ for some $m_{i,j} \in \mathbb{Z}$.

$l(w)$, the *length* of $w \in W$ is the length of the shortest word in the generators S which equals w.

$v \leq w$ in the weak order if there exists a sequence $s_{i_1}, \ldots, s_{i_j}$ of elements of S (not necessarily distinct) such that $v \cdot s_{i_1} \cdots s_{i_j} = w$ and $l(w) = l(v) + j$.

The symmetric group

Let $W = S_n$, the symmetric group on $[n] = \{1, \ldots, n\}$. Let S be the transpositions $s_i = (i, i+1)$. Then (W, S) is a Coxeter system with $m_{i,j} = 2$ if $|i-j| > 1$ and $m_{i,j} = 3$ if $|i-j| = 1$.

If v and w are elements of S_n written as words in $[n]$, then $v \leq w$ if w can be obtained from v by switching adjacent elements which are increasing.

Example: $[2134] < [2314] < [3214]$ in S_4.

Descent sets

Let $w \in W$. The *descent set* of w is

$$D(w) = \{s \in S : l(ws) < l(w).\}$$

For a subset $A \subseteq S$,

$$D(A) = \{w \in W : D(w) = A.\}$$

Symmetric group

Identify S and $[n-1]$. Then

$$D(w) = \{w : w(i) > w(i+1).\}$$

Example: $D[2134] = \{1\}, D[3214] = \{1,2\}$.

Definition 1 *Let* $A, B \subseteq S$. *Then* B *dominates* A *from above if there exists an injection* $\phi : D(A) \to D(B)$ *such that for all* $w \in D(A)$,

$$w \leq \phi(w).$$

Definition 2 *Let* $A, B \subseteq S$. *Then* A *dominates* B *from below if there exists an injection* $\phi : D(B) \to D(A)$ *such that for all* $w \in D(B)$,

$$w \geq \phi(w).$$

Example in S_4

$A = \{1\}, B = \{1, 2\}$.

$$\begin{array}{ccc} w & & \phi(w) \\ & & \\ [2134] & \Rightarrow & [3214] \\ [3124] & \Rightarrow & [4312] \\ [4123] & \Rightarrow & [4213] \end{array}$$

ϕ demonstrates that B dominates A from above.

ϕ^{-1} demonstrates that A dominates B from below.

Easy to show that if B dominates A from above (or below) in S_n, then this also holds in all S_m, $m \geq n$.

Theorem 3 *If B dominates A from above and Δ is*

- *The order complex of a geometric lattice ($[W = S_n]$ Nyman, S.)*

- *A finite building $[W =$ assoc. Coxeter group] (S)*

- *The order complex of the face poset of a CM complex ($[W = S_n]$ Schweig)*

then $h_A \leq h_B$.

Theorem 4 *If Δ is the order complex of a distributive lattice and A dominates B from below, then $h_B \leq h_A$.*

This follows easily from the usual S_n EL-labeling of the poset.

There is no known counterexample to the converse: If $h_B \leq h_A$ for all distributive lattices, then A dominates B from below.

Partial converse:

If $h_B \leq h_A$ for all distributive lattices, then $B \supseteq A$.

If A dominates B from below, then $B \supseteq A$.

Distributive lattice example continued

Recall in our example:

$$
\begin{array}{llll}
f_{\emptyset}=1 & h_{\emptyset}=1 & & \\
f_{\{1\}}=3 & f_{\{1,2\}}=7 & h_{\{1\}}=2 & h_{\{1,2\}}=1 \\
f_{\{2\}}=4 & f_{\{1,3\}}=9 & h_{\{2\}}=3 & h_{\{1,3\}}=3 \\
f_{\{3\}}=4 & f_{\{2,3\}}=8 & h_{\{3\}}=3 & h_{\{2,3\}}=1 \\
f_{\{4\}}=3 & f_{\{1,4\}}=8 & h_{\{4\}}=2 & h_{\{1,4\}}=3 \\
 & f_{\{2,4\}}=9 & & h_{\{2,4\}}=3 \\
 & f_{\{3,4\}}=7 & & h_{\{3,4\}}=1
\end{array}
$$

We also saw that $\{1\}$ dominates $\{1,2\}$ from below in S_5. Other pairs (A,B) with A dominating B from below in S_5 are

$$(\{2\},\{2,3\}),(\{3\},\{2,3\}),(\{4\},\{3,4\}).$$

Main Problem

Problem 1 *Given* (W, S) *determine when* B *dominates* A *from above.*

Theorem 5 *(E. Chong, 2009)*

- *If B dominates A from above, then $A \subseteq B$.*

- *Suppose s commutes with all $t \in A$. Then $A \cup \{s\}$ dominates A from above.*

Products

Suppose (W_1, S_1) and (W_2, S_2) are two finite Coxeter systems, B_1 dominates A_1 from above in W_1 and B_2 dominates A_2 in (W_2, S_2). Then it is easy to see that $B_1 \times B_2$ dominates $A_1 \times A_2$ from above.

However, the converse is false:

In $S_n \times S_m$ we see that $B \times [m-1]$ dominates $\emptyset \times A$ from above whenever $|D(B)| > |D(A)|$.

Symmetric group

For $w \in S_n$ let $R(w)$ be w written in reverse. Equivalently,

$$R(w) = w \cdot [n\ n-1 \ldots 321].$$

$D(R(w)) = \{i : n-1-i \notin D(w).\}$

Define $R(A)$ to be the common descent set of all permutations in $D(A)$.

Conjecture 1 *(Nyman - S.) If $A \subseteq R(A)$, then $R(A)$ dominates A from above.*

Verified through S_{10} by computer, and for all n with $|A| = 1$. (T. DeVries)

C. Boulet observed that there are no known counterexamples in S_n known for

$$A \subseteq B, \ |D(A)| \leq |D(B)| \rightarrow$$

$$B \text{ dominates } A \text{ from above.}$$

Other than some data generated by computer, almost nothing is known about the other irreducible finite Coxeter groups.

Variations

Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_l$ be subsets of S with $A_i \neq A_j$ and $B_i \neq B_j$ for $i \neq j$.

If there exists an injective map

$\phi : D(A_1) \cup \cdots \cup D(A_m) \to D(B_1) \cup \cdots \cup D(B_l)$

such that $w \leq \phi(w)$ for all w, then

$$h_{A_1} + \cdots + h_{A_m} \leq h_{B_1} + \cdots + h_{B_l}.$$

(Geometric lattices, finite buildings, face posets of CM complexes)

Example

$A_1 = \{2\}, A_2 = \{3\}, B_1 = \{2,3\}, B_2 = \{1,3\}$

w	$\phi(w)$
[1243]	[2143]
[1342]	[1432]
[2341]	[2431]
[1324]	[3142]
[1423]	[4132]
[2314]	[3241]
[2413]	[4231]
[3412]	[3421]

This ϕ shows that $h_{\{2\}} + h_{\{3\}} \leq h_{\{2,3\}} + h_{\{1,3\}}$ for rank 4 geometric lattices and the reverse inequality for rank 4 distributive lattices.

Combined with the previous example gives

$$h_1 \leq h_2.$$

Theorem 6 *(Nyman, S.) Using these methods it is possible to explain over 50% of the inequalities*

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}.$$

Earliest known form of the problem:

Problem 2 *Find a bijection ϕ from elements with i-descents to elements with $n-i$ descents, where $n = |S|$, such that $w \leq \phi(w)$ for all w.*

$S_n, i = 1$: P. Edelman (unpublished $\sim$ '99)

$S_n, i = 1, 2$ and $B_n, i = 1$ (Yessenov $\sim$ '05)

$S_n,\ n \leq 9; B_n,\ n \leq 6$. (DeVries, '05 via computer)

Note: It is not instantaneously obvious how this problem behaves under product.

Face posets

All linear flag h-vector inequalities for order complexes of face posets of CM complexes are 'known'.

Theorem 7 *(Stanley) Any linear inequality on all f-vectors of CM complexes is a consequence of $h_i \geq 0$ for all i.*

Suppose Δ is a $(d-1)$-dimensional CM complex and $F(\Delta)$ is its face poset. Then for any A, $h_A(F(\Delta))$ can be written in terms of the h-vector of Δ.

Let Δ be a $(d-1)$-dimensional complex.

Exercise 1 *(Stanley)*

$$h_A = \sum_{i=0}^{d} c_{A,i} \cdot h_i,$$

where

$$c_{A,i} = |\{w \in S_{d+1} : D(w) = A, w(d+1) = d-i+1.\}|$$

In particular, h_A is a nonnegative linear combination of the h_i and $h_A \leq h_B$ for all face posets of CM complexes if and only if

$$c_{A,i} \leq c_{B,i} \text{ for all } i.$$

Example

$d = 4, B = \{1, 2\}, A = \{1\}$

B	A
[32145]	[21345]
[42135]	[31245]
[43125]	[41235]
[52134]	[51234]
[53124]	
[54123]	

$$c_{B,0} = c_{A,0} = 3$$

$$c_{B,1} = 2,\ c_{A,1} = 1$$

$$c_{B,2} = 1,\ c_{A,2} = 0$$

So what is the point?

Easier conjecture: If $A \subseteq R(A)$, then for all i,

$$c_{A,i} \leq c_{R(A),i}.$$

Other linear inequalities

Example:

In S_4 consider $D(\{1,3\})$ and $D(\{1\})$. For any subset $X \subseteq D(\{1\})$ then number of elements π in $D(\{1,3\})$ such that π is greater than some element $\sigma \in X$ is at least $\frac{5}{3}|X|$. Hence

$$\frac{5}{3}h_{\{1\}} \leq h_{\{1,3\}}.$$

(Rank 4 geometric lattices ...)