Balanced and Unbalanced Collections

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1. Balanced and Unbalanced Collections
   - Balanced Collections – Economic Equilibria
   - Unbalanced Collections - Quantum Field Theory
   - Poset structure of maximal unbalanced collections (Björner)

2. Hyperplane Arrangements and Unbalanced Collections
   - All-subset arrangements
   - Lower bounds on the number of unbalanced collections
   - Upper bounds on the number of unbalanced collections
   - Threshold collections and threshold functions

3. Some Questions
For \( S \subseteq [n] = \{1, 2, \ldots, n\} \), let \( e_S := \sum_{i \in S} e_i \), where \( e_i = (0, \ldots, 1, \ldots, 0) \) is the \( i^{th} \) unit vector in \( \mathbb{R}^n \).
For $S \subseteq [n] = \{1, 2, \ldots, n\}$, let $e_S := \sum_{i \in S} e_i$, where $e_i = (0, \ldots, 1, \ldots, 0)$ is the $i^{th}$ unit vector in $\mathbb{R}^n$.

A collection $\mathcal{F} \subseteq 2^{[n]}$ is said to be balanced if

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for some $0 < \delta \leq 1$. 
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3) $\binom{[n]}{k}$ in $[n]$
Balanced collections were introduced 50 years ago by Lloyd Shapley (Nobel Prize in Economics, 2012) to characterize when cooperative games (with transferable utility) were robust enough (so-called balanced games) to ensure that players could be paid enough to guarantee that no subset could do better by leaving the coalition of everyone.
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Your speaker spent many years trying to generalize this to the nontransferable utility case, with some but not complete success.
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*i.e.*, any $x \in \mathbb{R}^n$ with $\sum_{i \in [n]} x_i = v([n])$ is a possible outcome.
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The core of \( v \) is the set of outcomes for which no coalition \( S \subset [n] \) can do better for all its members:

\[
\left\{ x \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = v([n]), \quad \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset [n] \right\}
\]
For some games, the core may be empty:

Theorem (Shapley-Bondareva): A game \( v \) on \( [n] \) has a nonempty core \( \iff \) \( v \) is balanced: for every minimal balanced collection \( F \), if \( e[n] = \sum_{S \in F} \delta_S e_S \) then \( v([n]) \geq \sum_{S \in F} \delta_S v(S) \).

Theorem (Shapley-Shubik): A game \( v \) on \( [n] \) arises from an economic trading model with convex preferences \( \iff \) for each \( S \subseteq [n] \), the subgame \( v|_S \) on \( S \) is balanced (has a nonempty core).

Note: In the NTU case, where \( V(S) \) is a set in place of a number, \( \text{market} \Rightarrow \text{balanced} \Rightarrow \text{core nonempty} \) still holds (with inclusion and set sums) [Scarf], while the converse of the second (balanced \( \Rightarrow \) market) has been proved in many, but not all, cases [B, et al.].
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Basic linear alternative theorem: Either $\mathcal{F}$ is balanced or $\exists w \in \mathbb{R}^n$, with $\sum_{i \in [n]} w_i = 0$ and $\sum_{i \in S} w_i > 0$ for $S \in \mathcal{F}$.

Thus maximal unbalanced collections are the same as Björner's PSS (positive set sum) systems. We are interested in enumerating these collections.
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Unbalanced collections arise in thermal field theory
Applications to Physics

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thermal field theory = quantum field theory + statistical mechanics

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\begin{array}{cccccccc}
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 6 & 32 & 370 & 11,292 & 1,066,044 & 347,326,352 & 419,172,756,930 \\
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Driving in Sicily!
A few examples

For $n = 3$, the 6 maximal unbalanced collections are

$$\left\{ \{1, 2\}, \{1, 3\}, \{1\} \right\}, \left\{ \{1, 2\}, \{2, 3\}, \{2\} \right\}, \left\{ \{1, 3\}, \{2, 3\}, \{3\} \right\}$$

$$\left\{ \{2\}, \{3\}, \{2, 3\} \right\}, \left\{ \{1\}, \{3\}, \{1, 3\} \right\}, \left\{ \{1\}, \{2\}, \{1, 2\} \right\}$$

For weight vectors $w = (2, -1, -1)$ and $w = (-2, 1, 1)$. 

For $n = 4$, two of the 32 such collections are

$$\left\{ \{1\} \right\}, \left\{ \{1\}, \{2\} \right\}, \left\{ \{1\}, \{3\} \right\}, \left\{ \{1\}, \{4\} \right\}, \left\{ \{1\}, \{2\}, \{3\} \right\}, \left\{ \{1\}, \{2\}, \{4\} \right\}, \left\{ \{1\}, \{3\}, \{4\} \right\}$$
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- they always have $2^{n-1} - 1$ sets and rank $n - 2$ with $(n - 1)!$ maximal chains.
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- their \( f \)-vectors are all the **same**;
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- they always have $2^{n-1} - 1$ sets and rank $n - 2$ with $(n - 1)!$ maximal chains.
- their order complexes are always **shellable balls** with a single interior vertex
- their $f$-vectors are all the same; in fact, $h_i(\Delta(\mathcal{F}))$ is the number of permutations in $S_{n-1}$ with $i$ descents (classical Eulerian numbers).
The simplicial complex $\Delta(\mathcal{F})$

Examples:
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Examples: $n = 3$
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we get

Here both have $f(\Delta) = (7, 12, 6)$ and a single interior vertex.
Recall: $\mathcal{F} \subset 2^{[n]}$ is unbalanced $\iff$ 
$\exists w \in \mathbb{R}^n$, with $\sum_{i \in [n]} w_i = 0$ and $\sum_{i \in S} w_i > 0$ for $S \in \mathcal{F}$. 

Restricted all-subset arrangement in $\mathbb{R}^n$
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This defines a hyperplane arrangement in $\mathbb{R}^n$, actually on the hyperplane $H_0 := \{ x \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = 0 \}$ (the space of all possible $w$'s), called the **restricted all subsets arrangement**, with all the hyperplanes having normals $e_S$, $S \subset [n], S \neq \emptyset, [n]$. 
Restricted all-subset arrangement in $\mathbb{R}^n$

Recall: $\mathcal{F} \subset 2^{[n]}$ is unbalanced $\iff \exists w \in \mathbb{R}^n$, with $\sum_{i \in [n]} w_i = 0$ and $\sum_{i \in S} w_i > 0$ for $S \in \mathcal{F}$.

This defines a hyperplane arrangement in $\mathbb{R}^n$, actually on the hyperplane $H_0 := \{x \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = 0\}$ (the space of all possible $w$’s), called the restricted all subsets arrangement, with all the hyperplanes having normals $e_S$, $S \subset [n]$, $S \neq \emptyset, [n]$.

The maximal (full-dimensional) regions in this arrangement are in bijection with the maximal unbalanced collections in $2^{[n]}$. 
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All-subset arrangement in $\mathbb{R}^{n-1}$

Combinatorially equivalent to the restricted all-subset arrangement in $\mathbb{R}^n$ is the all-subset arrangement $\mathcal{A}_{n-1}$ in $\mathbb{R}^{n-1}$, consisting of all hyperplanes with normals $e_S$, $S \subseteq [n-1], S \neq \emptyset$. 
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Again, regions of $\mathcal{A}_{n-1}$ are in bijection with maximal unbalanced collections in $2^{[n]}$. 

Example: $n = 3$. The planes of $\mathcal{A}_2$ are $x_1 = 0$, $x_2 = 0$, $x_1 + x_2 = 0$, so $\mathcal{A}_2$ has 6 regions:
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$$(-1)^n \chi(\mathcal{A}_n, -1) = \sum_{x \in L_n} |\mu(0, x)| = \sum_{k=0}^{n} |w_k(L_n)|.$$
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Unfortunately, we don’t know $\chi(A_n, t)$. 
Consider the binary matroid $\mathcal{A}_n^2$ consisting of all subspaces spanned over the 2-element field $\mathbb{F}_2$ by all the nonzero elements of $\{0, 1\}^n$.\vspace{1cm}
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The identity map $\mathcal{A}_n \rightarrow \mathcal{A}_n^2$ is a rank-preserving weak map (inverse image of independent sets are independent), so by the theorem of Lucas

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The “binary all-subsets arrangement”

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$$|w_k(A_n)| \geq |w_k(A^{(2)}_n)|$$

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Theorem: The number of maximal unbalanced families in $[n]$, equivalently, the number of chambers of the arrangement $A_{n-1}$, is at least $\prod_{i=0}^{n-2} (2^i + 1)$. Thus the number of maximal unbalanced collections is more than

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This answers a question raised by the physicist T.S. Evans, who asked if the number of such collections exceeded \(n!\).
To give an upper bound, we consider the signature (degree sequence) of an unbalanced family $F$ in \[ n \]

$$\text{sig}(F) := (s_1, ..., s_n)$$

where $s_i = |\{ F \in F | i \in F \}|$.

$\text{sig}(\cdot)$ is injective over maximal unbalanced families. If $F$ is maximal, then all entries of $\text{sig}(F)$ have the same parity.

$|F| = 2^n - 1 - 1$ for maximal unbalanced families, so there are fewer than \(2^n - 1\) possible signatures.

Theorem: There are fewer than $2^{(n-1)^2}$ maximal unbalanced families in $[n]$.  

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• A collection of subsets $\mathcal{T} \subset 2^{[n]}$ is a threshold collection if there is a weight vector $w \in \mathbb{R}^n$ and $q \in \mathbb{R}$ so that

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Note: A Boolean function $f : \{0, 1\}^n \to \{0, 1\}$ is a threshold function iff there is a threshold collection $\mathcal{T}$ so that

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Threshold collections and threshold functions

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Thus $\{\text{unbalanced } \mathcal{T}\} \subset \{\text{0-threshold } \mathcal{T}\} \subset \{\text{threshold } \mathcal{T}\}$
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Theorem (Zuev, 1989): \( \log_2 E_n \sim (n - 1)^2 \) as \( n \to \infty \)
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• \( \mathcal{A}_n \) all-subset arrangement in \( \mathbb{R}^n \), consisting of all hyperplanes with normals \( e_S, S \subseteq [n], S \neq \emptyset \).

• \( E_n \) is the number of regions in \( \mathcal{A}_{n-1} \)

But the regions in \( \mathcal{A}_n \) also correspond to 0-threshold collections in \( 2^n \). Thus \( T_{n-1}^0 = E_n \) and so our bounds were already known. In fact:

Theorem (Zuev, 1989): \( \log_2 E_n \sim (n - 1)^2 \) as \( n \to \infty \)

The argument uses a theorem of Odlyzko on random \( \pm 1 \) vectors.
Further, one can describe threshold collections $\mathcal{T} \subset 2^{[n]}$ via

$$\exists (w, q) \in \mathbb{R}^{n+1} \text{ so that } S \in \mathcal{T} \iff \sum_{S} w_i + q > 0$$
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$$T_n < T^0_{n+1} = E_{n+2}$$

How much less is not known but should be.
Minimal balanced collections can be viewed as generalized partitions. Is there a nice poset structure for them?
Open questions

- Minimal balanced collections can be viewed as generalized partitions. Is there a nice poset structure for them?
- Determine $\chi(A_n, t)$ exactly for all $n$. Kamiya, Takemura and Terao have computed it for $n \leq 8$. 
- Is there some sort of resolution theory for weak maps that would enable this computation?
- The signature, and more generally, the degree sequence of graphs and threshold complexes, behaves like the coordinates for secondary polytopes given by Gel'fand, Kapranov and Zelevinski. Is there some relation here?
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Some references


[includes references to the economics/physics applications, in particular]


