Balanced and Unbalanced Collections

Louis J. Billera
Cornell University

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1. Balanced and Unbalanced Collections
   - Balanced Collections – Economic Equilibria
   - Unbalanced Collections - Quantum Field Theory
   - Poset structure of maximal unbalanced collections (Björner)
2. Hyperplane Arrangements and Unbalanced Collections
   - All-subset arrangements
   - Lower bounds on the number of unbalanced collections
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3. Some Questions

Balanced Collections

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Cooperative games and economic equilibria

Balanced collections were introduced 50 years ago by Lloyd Shapley (Nobel Prize in Economics, 2012) to characterize when cooperative games (with transferable utility) were robust enough (so-called balanced games) to ensure that players could be paid enough to guarantee that no subset could do better by leaving the coalition of everyone.

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Your speaker spent many years trying to generalize this to the nontransferable utility case, with some but not complete success.

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The core of v is the set of outcomes for which no coalition $S \subset [n]$ can do better for all its members:

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i\in[n]} x_i = v([n]),\;\; \sum_{i\in S} x_i \geq v(S) \text{ for all } S \subset [n] \right\}$$

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Note: In the NTU case, where $V(S)$ is a set in place of a number, market $\Rightarrow$ balanced $\Rightarrow$ core nonempty still holds (with inclusion and set sums) [Scarf], while the converse of the second (balanced $\Rightarrow$ market) has been proved in many, but not all, cases [B__, *et al.*].

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We are interested in enumerating these collections.

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Driving in Sicily!

A few examples

For $n = 3$, the 6 maximal unbalanced collections are

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For $n = 4$, two of the 32 such collections are

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- their order complexes are always shellable balls with a single interior vertex
- their f-vectors are all the same; in fact, $h_i(\Delta(\mathcal{F}))$ is the number of permutations in S_{n-1} with i descents (classical Eulerian numbers).

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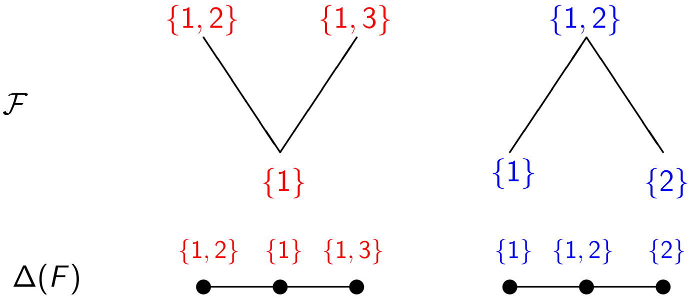

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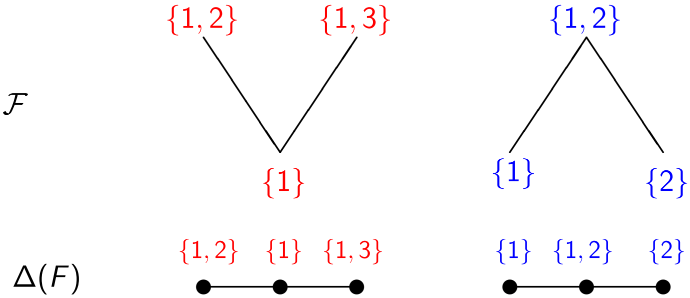

Note: both have $f(\Delta) = (3,2)$ and a unique interior vertex

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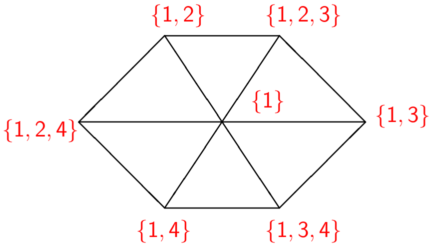

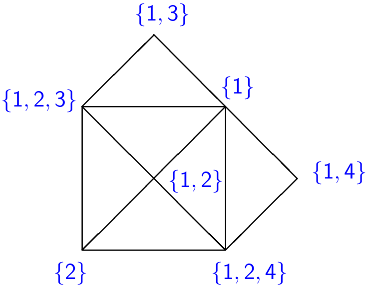

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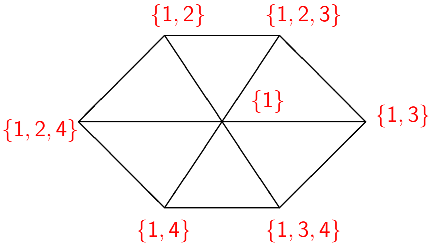

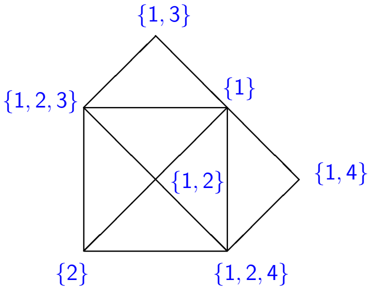

Here both have $f(\Delta) = (7, 12, 6)$ and a single interior vertex.

Restricted all-subset arrangement in $\mathbb{R}^n$

Recall: $\mathcal{F} \subset 2^{[n]}$ is unbalanced $\iff$
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The maximal (full-dimensional) regions in this arrangement are in bijection with the maximal unbalanced collections in $2^{[n]}$.

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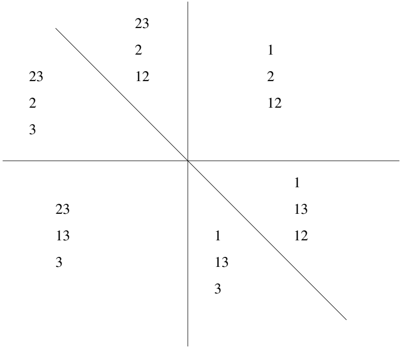

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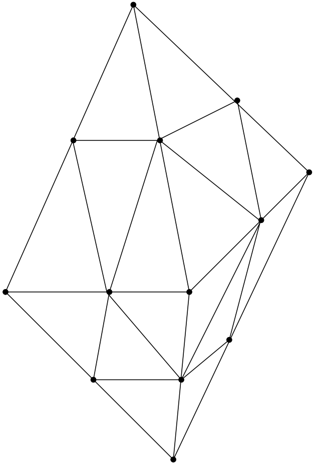

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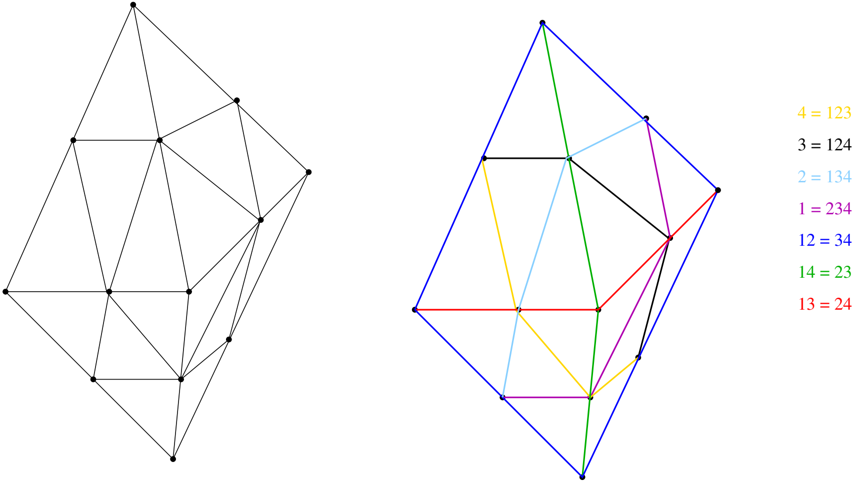

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Theorem: The number of maximal unbalanced families in $[n]$, equivalently, the number of chambers of the arrangement $\mathcal{A}_{n-1}$, is at least $\prod_{i=0}^{n-2}(2^i+1)$. Thus the number of maximal unbalanced collections is more than

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This answers a question raised by the physicist T.S. Evans, who asked if the number of such collections exceeded $n!$.

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Theorem: There are fewer than $2^{(n-1)^2}$ maximal unbalanced families in $[n]$.

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Threshold collections and threshold functions

- A collection of subsets $\mathcal{T} \subset 2^{[n]}$ is a threshold collection if there is a weight vector $w \in \mathbb{R}^n$ and $q \in \mathbb{R}$ so that

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Thus $\{$ unbalanced $\mathcal{T}\} \subset \{$ 0-threshold $\mathcal{T}\} \subset \{$ threshold $\mathcal{T}\}$

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The argument uses a theorem of Odlyzko on random ± 1 vectors.

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Further, one can describe threshold collections $\mathcal{T} \subset 2^{[n]}$ via

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- Is there some sort of resolution theory for weak maps that would enable this computation?
- The signature, and more generally, the degree sequence of graphs and threshold complexes, behaves like the coordinates for secondary polytopes given by Gel'fand, Kapranov and Zelevinski. Is there some relation here?

Some references

L. Billera, J. Moore, C. Moraites, Y. Wang, C. Williams, Maximal unbalanced families, arXiv:1209.2309 [math.CO], 11 Sep 2012.

[includes references to the economics/physics applications, in particular]

T. S. Evans, What is being calculated with Thermal Field Theory?, In A. Astbury, B. A. Campbell, W. Israel, F. C. Khanna, D. Page, and J. L. Pinfold, editors, *Particle Physics and Cosmology - Proceedings of the Ninth Lake Louise Winter Institute*, pages 343–352. World Scientific, 1995.

Saburo Muroga, *Threshold Logic and its Applications, Wiley, 1971.*

A. M. Odlyzko, On subspaces spanned by random ± 1 *vectors*, JCT A **47** (1988), 124-133.

Yu. A. Zuev, Asymptotics of the logarithm of the number of threshold functions of the algebra of logic, Soviet Math. Dokl. **39** (1989), 512-513.