Quasisymmetric refinements of Schur functions

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Compositions and partitions

A composition $\alpha_1 \ldots \alpha_k$ of n is a list of positive integers whose sum is n: $2213 \vDash 8$.

A composition is a partition if $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k > 0$: $3221 \vdash 8$.

Any composition determines a partition: $\lambda(2213) = 3221$.

$\alpha = \alpha_1 \ldots \alpha_k$ is a coarsening of $\beta = \beta_1 \ldots \beta_l$ (β is a refinement of α) if

$$\underbrace{\beta_1 + \ldots + \beta_i}_{\alpha_1} \underbrace{\beta_{i+1} + \ldots + \beta_j}_{\alpha_2} \ldots \underbrace{\beta_m + \ldots + \beta_l}_{\alpha_k}$$

is true: $53 \succcurlyeq 2213$.

Quasisymmetric functions

Let $QSym$ be the algebra of quasisymmetric functions

$$QSym := QSym_0 \oplus QSym_1 \oplus \cdots \subset \mathbb{Q}[x_1, x_2, \ldots]$$

$$QSym_n := \mathrm{span}_{\mathbb{Q}}\{M_\alpha \mid \alpha = \alpha_1 \ldots \alpha_k \vDash n\} = \mathrm{span}_{\mathbb{Q}}\{F_\alpha \mid \alpha \vDash n\}$$

$$M_\alpha := \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} \qquad F_\alpha = \sum_{\alpha \succcurlyeq \beta} M_\beta$$

Example $M_{121} = \sum_{i_1<i_2<i_3} x_{i_1}^1 x_{i_2}^2 x_{i_3}^1$, $F_{121} = M_{121} + M_{1111}$

Remark $Sym \hookrightarrow QSym$ via $m_\lambda = \sum_{\lambda(\alpha)=\lambda} M_\alpha$.

Why quasisymmetric functions?

• Generating functions for P-partitions, posets, matroids (Gessel 83, Ehrenborg 96, Stembridge 97, Petersen 07, Luoto 09, Billera-Jia-Reiner 09).

• Combinatorial Hopf algebras (Ehrenborg 96, Aguiar-Bergeron-Sottile 06).

• Dual to cd-index (Billera-Hsiao-vW 03).

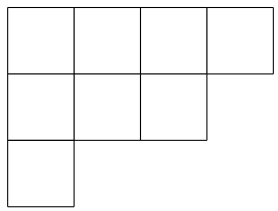
• Random walks (Stanley 01, Hsiao-Hersh 09).

• Simplify Macdonald polys (Haglund-Luoto-Mason-vW 09).

• Other types, coloured, shifted (Billey-Haiman 95, Hsiao-Petersen 10).

Diagrams and tableaux

The diagram $\lambda = \lambda_1 \geq \ldots \geq \lambda_k > 0$ is the array of boxes with λ_i boxes in row i from the top.



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A (standard) reverse tableau T of shape λ is a filling of λ with (each first n) $1, 2, 3, \ldots$ so rows weakly decrease and columns strictly decrease.

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8	7	3	1
6	4	2	
5			

431

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Composition diagrams and tableaux

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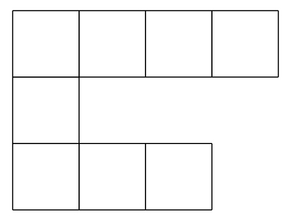

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A (standard) composition tableau of shape α is a filling of α with (each first n) $1, 2, 3, \dots$ such that

Rules for composition tableaux

- First column entries strictly increase top to bottom.
- Rows weakly decrease left to right.
- If $b \leq c$ then $b < a$.

Example

c	a
	b

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6			
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Quasisymmetric Schur functions

If $x^T := x_1^{\#1s} x_2^{\#2s} x_3^{\#3s} \ldots$ then $QSym_n = \text{span}_{\mathbb{Q}}\{\mathcal{S}_\alpha \mid \alpha \vDash n\}$ where

$$\mathcal{S}_\alpha = \sum_{T \in CT(\alpha)} x^T$$

Example

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Example $\mathcal{S}_{12} = x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3^2$ from

1	
2	2

1	
3	2

1	
3	3

2	
3	3

$s_\lambda = \sum_{\lambda(\alpha)=\lambda} \mathcal{S}_\alpha$ as $m_\lambda = \sum_{\lambda(\alpha)=\lambda} M_\alpha$.

Quasisymmetric Kostka numbers

For $\lambda \vdash n$

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$$

where $K_{\lambda\mu}$ = number of reverse tableaux T of shape λ and μ_1 1s, μ_2 2s, . . .

For $\alpha \vDash n$

$$\mathcal{S}_\alpha = \sum_\beta K_{\alpha\beta} M_\beta$$

where $K_{\alpha\beta}$ = number of composition tableaux T of shape α and β_1 1s, β_2 2s, . . .

Young's lattice: $\mathcal{L}_Y$

Partial order on partitions with covers

- add 1 at end: $211 < 2111$
- add 1 to leftmost part of size: $211 < 221$, $211 < 311$.

saturated chains in $\mathcal{L}_Y$ from μ to λ $\leftrightarrow$ standard skew RT shape λ/μ

Example

$32 < 321 < 331 < 431 \leftrightarrow$

•	•	•	1
•	•	2	
3			

Composition poset: $\mathcal{L}_C$

Partial order on compositions with covers

- add 1 at start: $121 < 1121$
- add 1 to leftmost part of size: $121 < 221$, $121 < 131$.

saturated chains in $\mathcal{L}_C$ from α to β	$\leftrightarrow$	standard skew CT shape $\beta//\alpha$

Example

$23 < 123 < 133 < 143 \leftrightarrow$

3			
•	•	2	1
•	•	•	

Descents and sets

T standard (skew) tableau, $Des(T) = \{i \mid i+1$ weakly east $\}$:

8	7	3	1
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5			

composition $\alpha_1 \ldots \alpha_k \vDash n \leftrightarrow$ subset $\{i_1, \ldots, i_{k-1}\} \subseteq [n-1]$

$\beta \qquad 2312 \vDash 8 \leftrightarrow \{2, 5, 6\} \subseteq [7] \qquad Set(\beta)$

Quasisymmetric skew Schur functions

Skew Schur functions

$$s_{\lambda/\mu} = \sum F_\delta = \sum_\nu c^\lambda_{\mu\nu} s_\nu \qquad c^\lambda_{\mu\nu} : \text{+ve integers}$$

where $Set(\delta) = Des(T)$, $T \in SRT(\lambda/\mu)$.

Quasisymmetric skew Schur functions

$$\mathcal{S}_{\gamma//\beta} = \sum F_\delta = \sum_\alpha C^\gamma_{\alpha\beta} \mathcal{S}_\alpha \qquad C^\gamma_{\alpha\beta} : \text{+ve integers}$$

where $Set(\delta) = Des(T)$, $T \in SCT(\gamma//\beta)$.

If $\lambda(\alpha) = \mu, \lambda(\beta) = \nu$ then $c^\lambda_{\mu\nu} = \sum_{\lambda(\gamma)=\lambda} C^\gamma_{\alpha\beta}$

What other Schur properties do $\mathcal{S}_\alpha$ have?

- $\mathbb{Z}$-basis for $QSym$. Expression in F_β. ✔

- Quasisymmetric Pieri, LR rules. ✔

- Involution gives row strict versions (Mason-Remmel 10, Ferreira 11) ✔

- Confirmed $QSym$ over Sym has a stable basis (Lauve-Mason 10).

"Just switch partition to composition"

Further properties?

Other properties:

- Jacobi-Trudi, Giambelli (quasi-) determinantal formulae?
- Representation theoretic interpretation from F_β?
- $\mathcal{L}_\mathcal{C}$ properties?
- Normal or Kronecker (inner) product?

Other applications:

- Quasisymmetric Macdonald polynomials?
- Skew Macdonald polynomials?
- Product of Schubert polynomials?
- Impact on $QSym$ of different types?

Link to NC Schurs of Fomin and Greene

P graded edge labelled poset, labels $(B, <)$. For $x \in P$

$$x.\mathbf{h}_k = \sum_{\omega} end(\omega)$$

$$\omega : x \xrightarrow{b_1} x_1 \xrightarrow{b_2} \cdots \xrightarrow{b_k} x_k = end(\omega)$$

for saturated ω, $b_1 \leq b_2 \leq \cdots \leq b_k \in B$.

For $[x, y]$ of P

$$K_{[x,y]} = \sum_{\alpha} \langle x.\mathbf{h}_\alpha, y \rangle M_\alpha \qquad \langle\ ,\ \rangle = \delta_{ij}$$

Example Skew Schur functions, Stanley symmetric functions, NC Schurs Fomin+Greene (Bergeron-Mykytiuk-Sottile-vW 00).

A new example

Let $\mathcal{L}'_C$ be the dual poset of $\mathcal{L}_C$ edges labelled

$$x \overset{(-col,-row)}{\longrightarrow} \tilde{x}$$

and $(i, j) < (k, \ell)$ iff $i < k$ or ($i = k = -1$ and $j > \ell$) or ($i = k < -1$ and $j < \ell$).

Then

$$K_{[\beta,\alpha]} = \mathcal{S}_{\beta//\alpha}.$$

Link to NC Schurs of Rosas and Sagan

A set composition of $[n] = \{1, \ldots, n\}$ is an ordered partitioning of $[n]$: $\Phi = 36/489/2/157 \models [9]$ with underlying composition $\alpha(\Phi) = 2313$.

A set partition of $[n]$ reorders by least element: $\tilde{\Phi} = 157/2/36/489 \vdash [9]$ with underlying partition $\lambda(\Phi) = 3321$.

Symmetric functions in noncommuting variables

(Wolfe 36, Rosas-Sagan 06)

$$NCSym := NCSym_0 \oplus NCSym_1 \oplus \cdots \subset \mathbb{Q}\langle x_1, x_2, \ldots \rangle$$

where

$$NCSym_n := \text{span}_{\mathbb{Q}}\{\mathbf{m}_\pi \mid \pi \vdash [n]\}$$

$$\mathbf{m}_\pi := \sum x_{i_1} x_{i_2} \cdots x_{i_n} \text{ and } i_j = i_k \text{ iff } j, k \in \pi_m$$

Example $\mathbf{m}_{13/2} = x_1x_2x_1 + x_2x_1x_2 + x_1x_3x_1 + x_3x_1x_3 \ldots$

NC Schurs of Rosas and Sagan

For $T \in RT(\lambda)$ let $\dot{T}$ have 1 entry with k dots $k = 1, 2, 3 \ldots$ then

$$S_\lambda^{RS} = \sum_{T \in RT(\lambda)} x^{\dot{T}} = \sum_\mu \mu! K_{\lambda\mu} \sum_{\lambda(\pi)=\mu} \mathbf{m}_\pi$$

where $x^{\dot{T}}$ = monomial x_i in position j if T has i with j dots.

Example

$$\begin{array}{|c|c|}\hline \dot{2} & \dddot{1} \\ \hline \ddot{3} & \\ \cline{1-1} \end{array} \qquad \rightsquigarrow \qquad x_2 x_3 x_1$$

Quasisymmetric functions in noncommuting variables

(Aguiar-Majahan 06, Bergeron-Zabrocki 09)

$$NCSym \subset NCQSym := NCQSym_0 \oplus NCQSym_1 \oplus \cdots \subset \mathbb{Q}\langle x_1, x_2, \ldots\rangle$$

where

$$NCQSym_n := \mathrm{span}_{\mathbb{Q}}\{\mathbf{M}_\Pi \mid \Pi \vDash [n]\}$$

$$\mathbf{M}_\Pi := \sum x_{i_1} x_{i_2} \cdots x_{i_n}$$

- $i_j = i_k$ iff $j, k \in \Pi_m$
- $i_j < i_k$ iff $j \in \Pi_{m_1}$ $k \in \Pi_{m_2}$ and $m_1 < m_2$.

Example $\mathbf{M}_{2/13} = x_2 x_1 x_2 + x_3 x_1 x_3 \ldots$

NC quasisymmetric Schurs

Let

$$S_\alpha^{RS} = \sum_{T \in CT(\alpha)} x^{\dot{T}} = \sum_\beta \beta! K_{\alpha\beta} \sum_{\alpha(\Pi)=\beta} \mathbf{M}_\Pi$$

Furthermore

$$\begin{array}{ccc} & S_\lambda^{RS} = & \sum_{\lambda(\alpha)=\lambda} S_\alpha^{RS} \\ Rosas{-}Sagan \rightsquigarrow & \chi \big\downarrow & \big\downarrow \chi \\ & n!\; s_\lambda = & n! \sum_{\lambda(\alpha)=\lambda} \mathcal{S}_\alpha \end{array}$$

Link to free Schurs of Poirier and Reutenauer (95)

$$PR := \text{span}_{\mathbb{Q}}\{T \mid T \in SRT\}$$

If $T_1 \in SRT(\mu)$ and $T_2 \in SRT(\nu)$ then $T_1 * T_2 = \sum_{SRT} T$ where

- $T \mid_{\mu} = T_1 + |\nu|$
- $rect(T \backslash \mu) = T_2$.

Example $\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} * \begin{array}{|c|}\hline 1 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 3 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} + \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} 1 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$

Bijection (Mason 06): $\rho : SCT \to SRT$

The map takes

5	4	3	1
6			
8	7	2	

$\mapsto$

8	7	3	1
6	4	2	
5			

Connection: $\phi : PR \to QSym^*$

If $\phi(T) = \mathcal{S}^*_\alpha$ where $\rho^{-1}(T) \in SCT(\alpha)$ then

$$\phi(T_1 * T_2) = \phi(T_2) \star \phi(T_1).$$

Further reading

• Skew quasisymmetric Schur functions and noncommutative Schur functions (with Bessenrodt and Luoto), Adv. Math., 226:4492–4532 (2011) .

• Refinements of the Littlewood-Richardson rule (with Haglund, Luoto and Mason), Trans. Amer. Math. Soc. 363:1665–1686 (2011).

• Quasisymmetric Schur functions (with Haglund, Luoto and Mason), J. Combin. Theory Ser. A 118: 463–490 (2011).

Thank you!