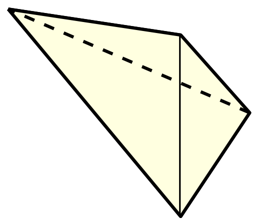
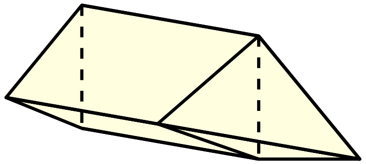
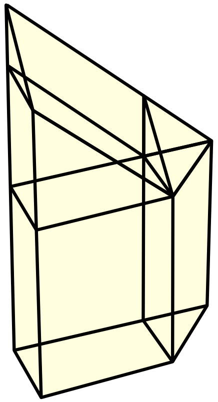
Flow polytopes in combinatorics and algebra

Karola Mészáros
Cornell University

Triangle Lectures in Combinatorics

March 24, 2018

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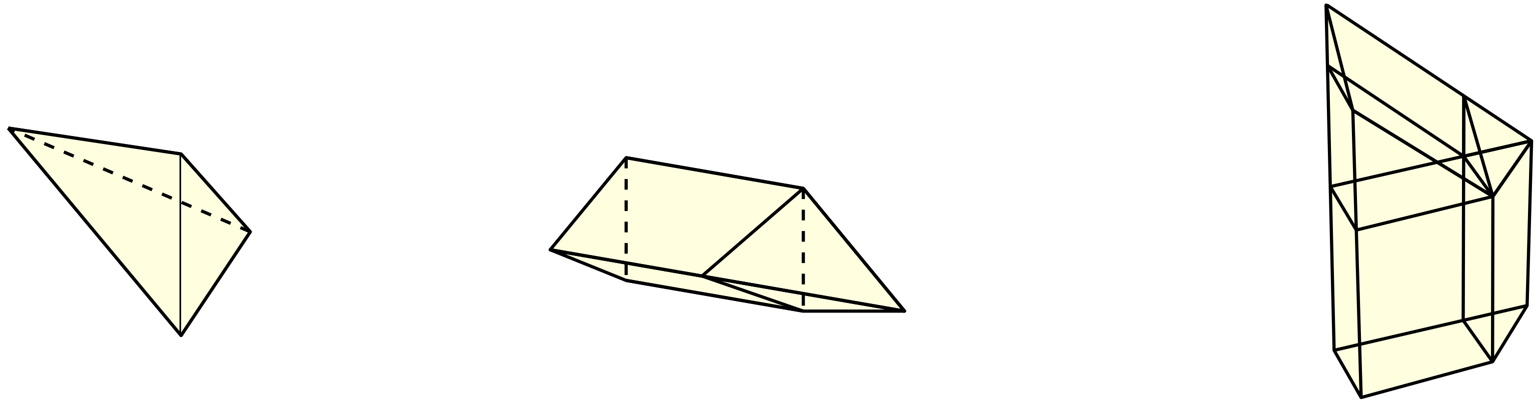

Thanks to Alejandro Morales for making a subset of the slides!

Volume and discrete volume

P a polytope in $\mathbb{R}^N$ with integral vertices

$\mathrm{vol}(P)$ is normalized volume with respect to underlying lattice

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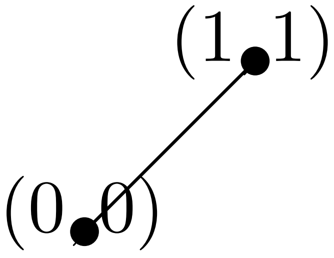

$\mathrm{vol}(P) = 1$

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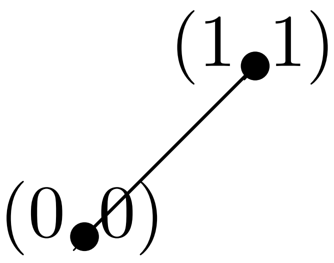

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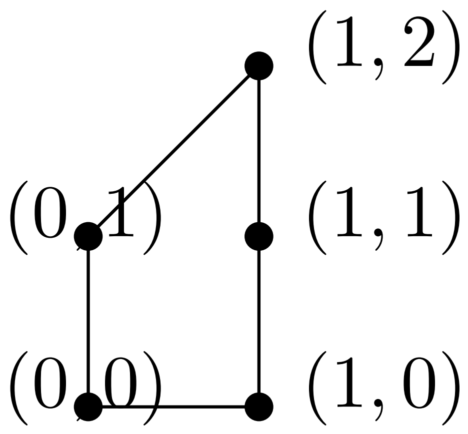

$\mathrm{vol}(P) = 3$

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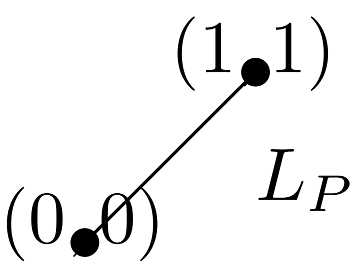

$$L_P(t) = t + 1$$

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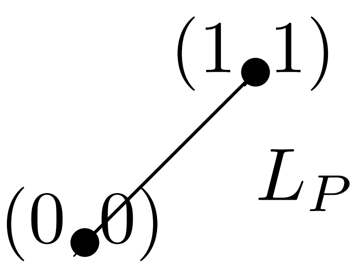

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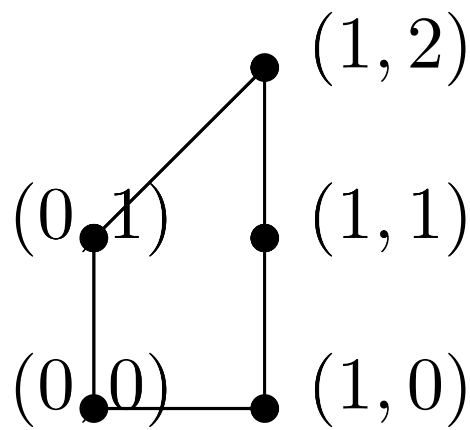

$L_P(t) = \frac{3}{2}t^2 + \frac{5}{2}t + 1$

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volume and number of lattice points of P are related:

$\mathrm{vol}(P)/\dim(P)! =$ leading coefficient $L_P(t)$

Flow polytopes

G directed graph on $n+1$ vertices

$\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ netflow

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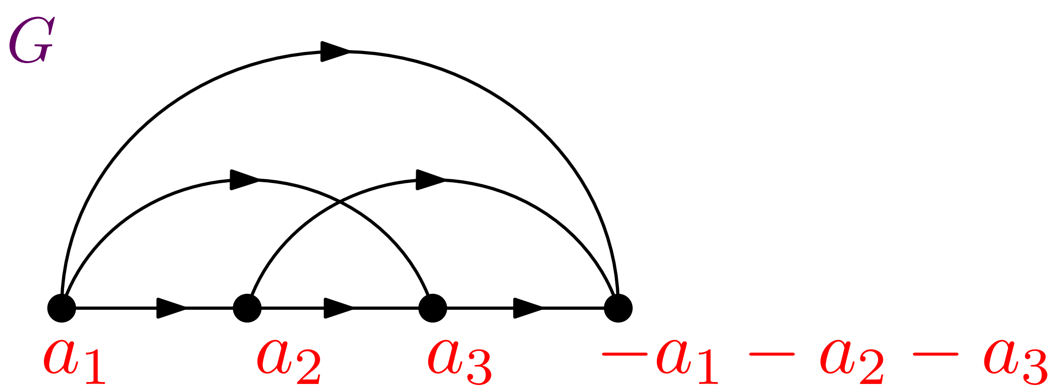

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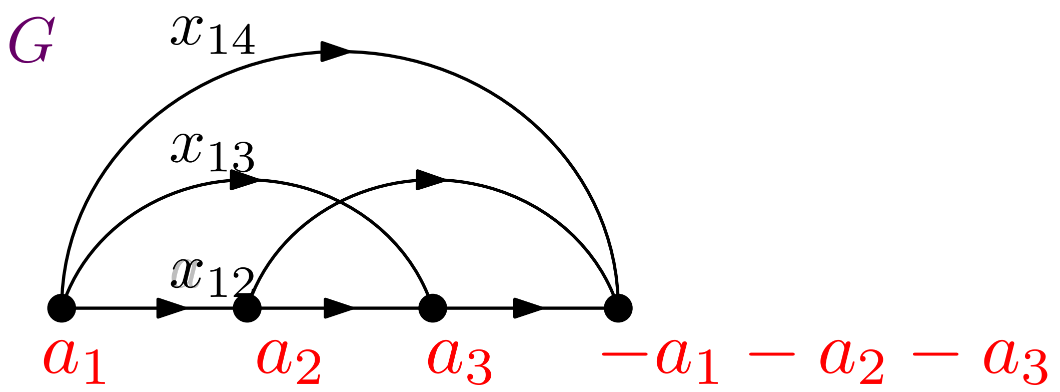

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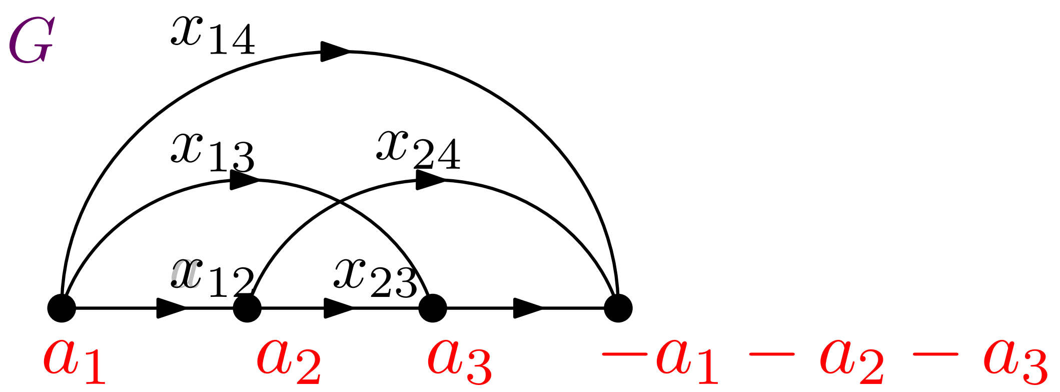

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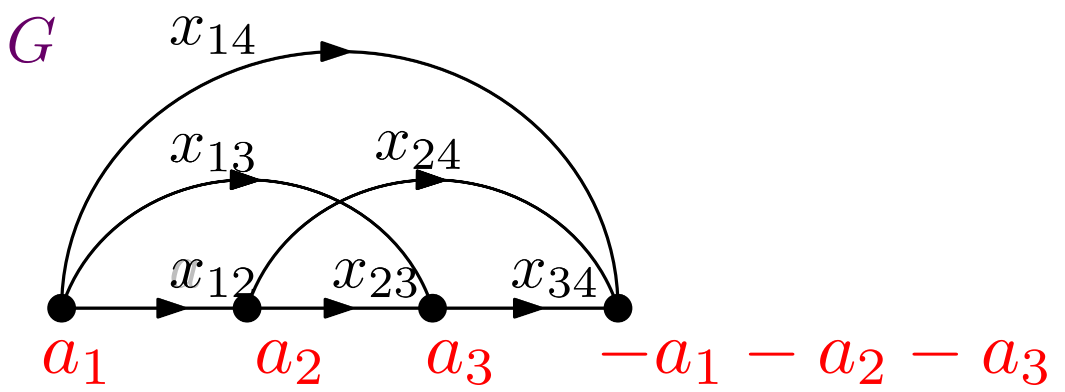

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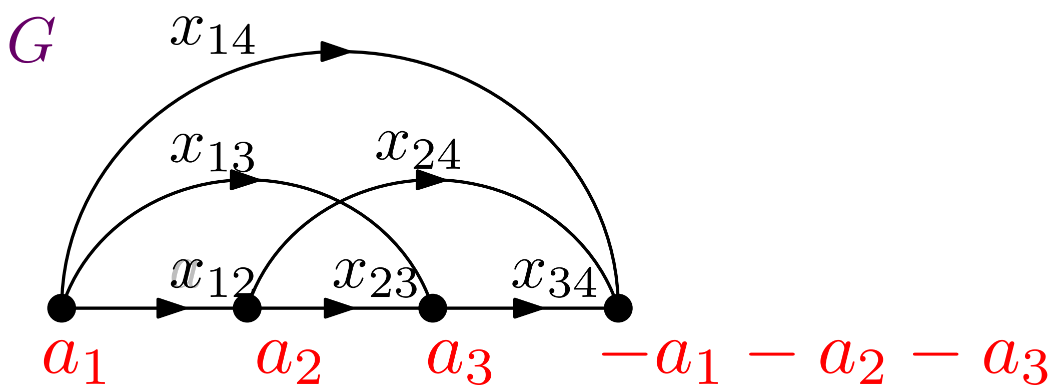

Lattice points of $\mathcal{F}_G(\mathbf{a})$ are integral flows on G with netflow $\mathbf{a}$.
Let $K_G(\mathbf{a}) := L_{\mathcal{F}_G(\mathbf{a})}(1)$.

Kostant partition function

When G is complete graph k_{n+1}, $K_{k_{n+1}}(\mathbf{a})$ is called the **Kostant partition function**.

$K_{k_{n+1}}(\mathbf{a}) = \#$ of ways of writing $\mathbf{a}$ as an $\mathbb{N}$-combination of vectors $e_i - e_j$, $1 \leq i < j \leq n+1$

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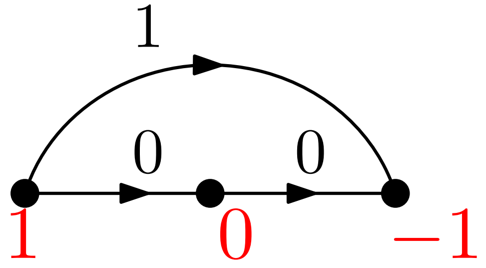
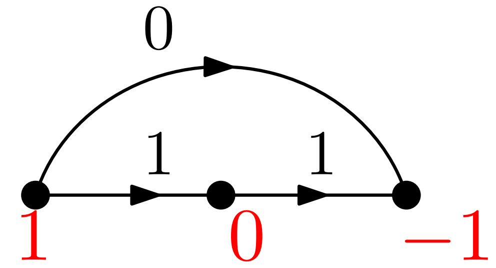


$(1, 0, -1) = e_1 - e_3$



$(1, 0, -1) = (e_1 - e_2) + (e_2 - e_3)$

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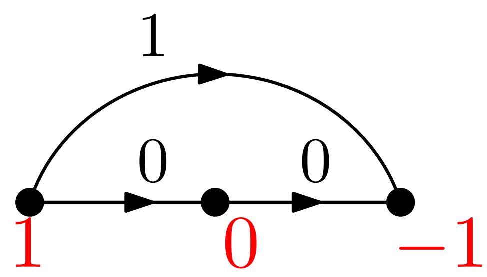

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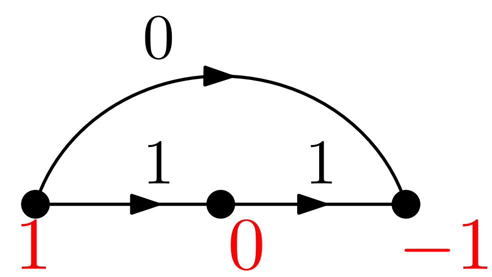

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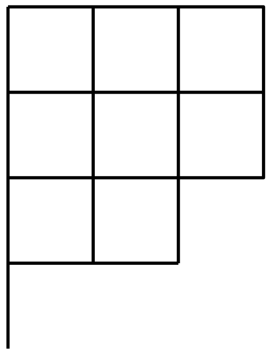
Formulas for **Kostka numbers** and **Littlewood-Richardson coefficients** in terms of $K_{k_{n+1}}(\mathbf{a})$.

Kostka numbers

$n = 4,\ \lambda = (3, 3, 2, 0),\ \mu = (2, 2, 2, 2)$



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4	4	

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Kostant's weight multiplicity formula:

$$K_{\lambda,\mu} = \sum_{w \in S_n} \operatorname{sgn}(w) K_{k_n}(w(\lambda+\rho) - (\mu+\rho)),$$

where $\rho = (n-1, n-2, \ldots, 1, 0)$.

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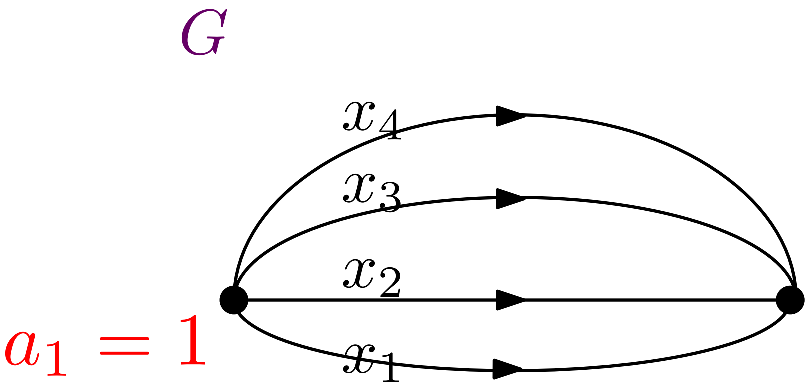
Examples of flow polytopes

$$\mathcal{F}_G(\mathbf{a}) = \{\text{flows } x(\epsilon) \in \mathbb{R}_{\geq 0},\ \epsilon \in E(G) \mid \text{netflow}(i) = a_i\}$$

Example

$$x_1 + x_2 + x_3 + x_4 = 1$$



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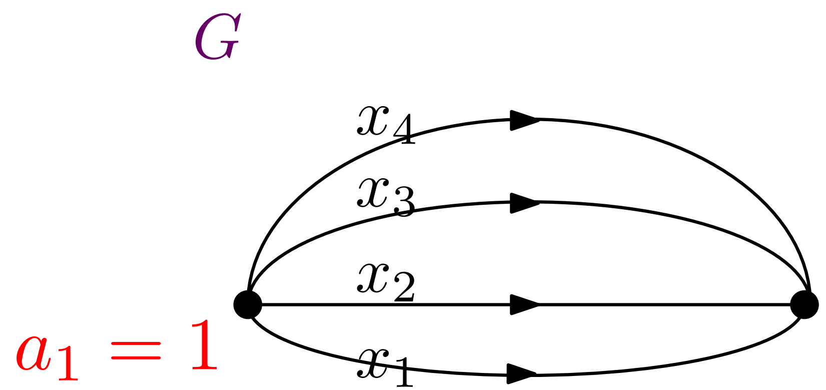
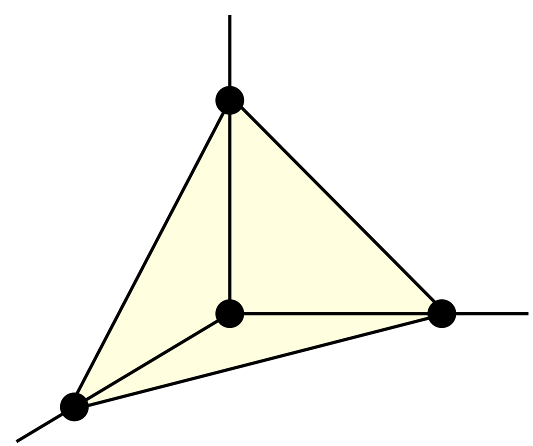


$\mathcal{F}_G(\mathbf{a})$ is a simplex



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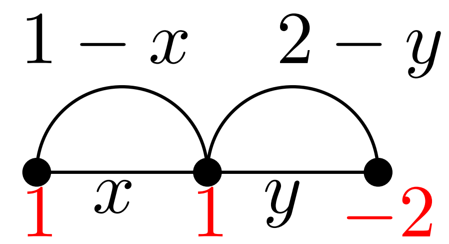

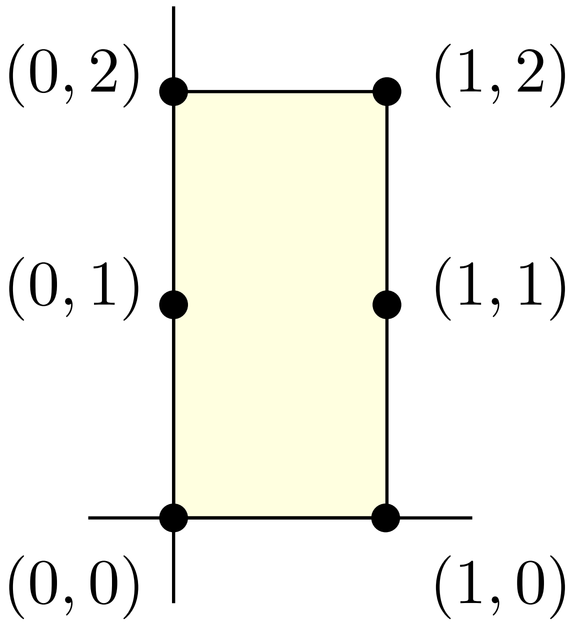

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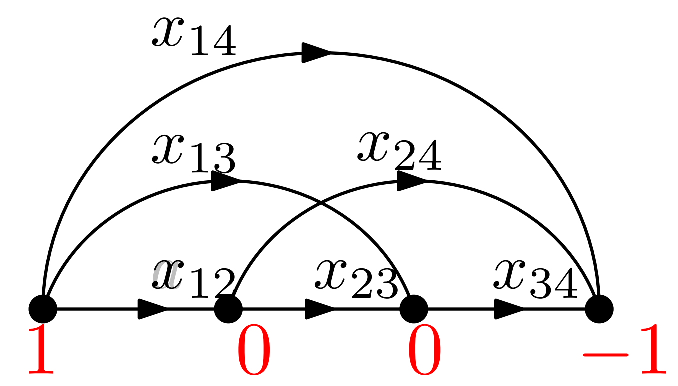
$\mathbf{a} = (1, 0, \ldots, 0, -1)$



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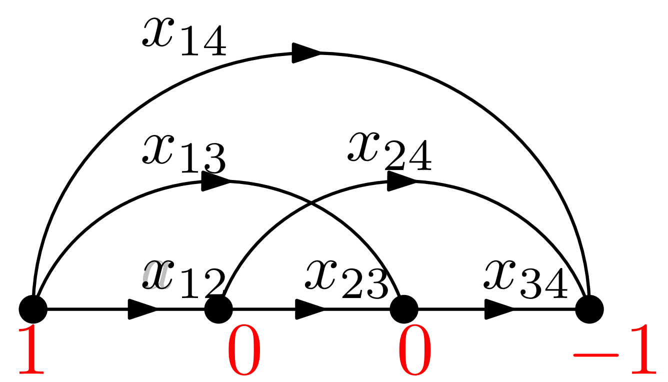

$\mathcal{F}_{k_{n+1}}(1, 0, \ldots, 0, -1)$ is called the **Chan-Robbins-Yuen** (CRY_n) polytope

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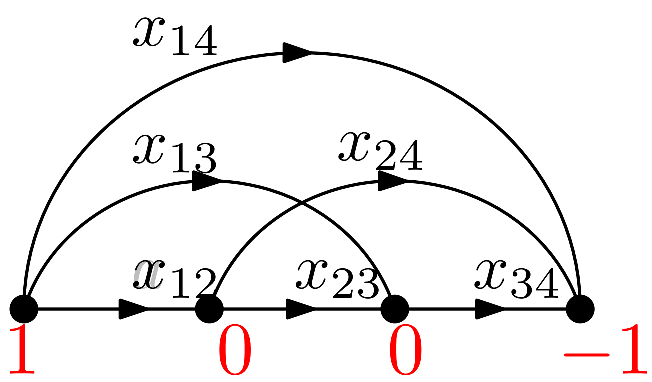

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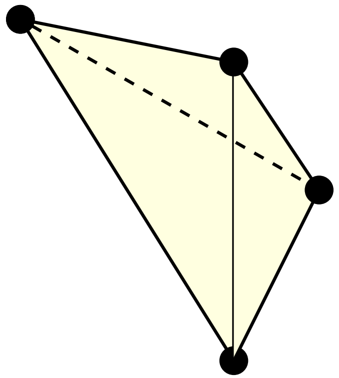

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(conjecture Chan-Robbins-Yuen 99)

- $v_n = C_1 \cdots C_{n-2}$ (Zeilberger 99)

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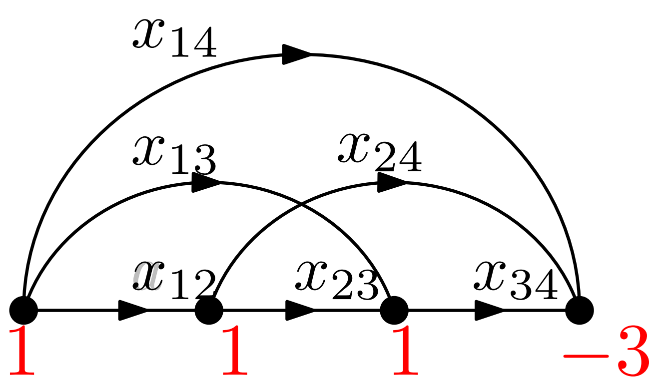

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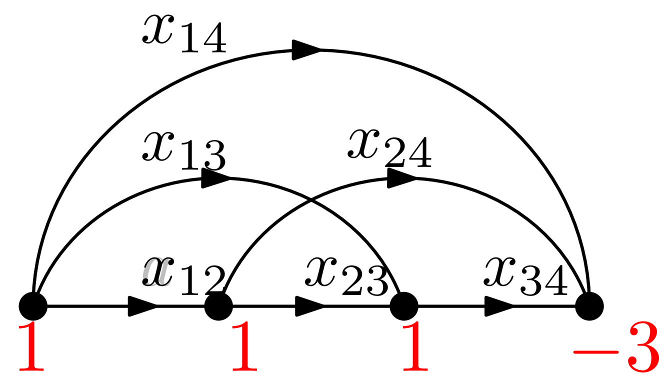

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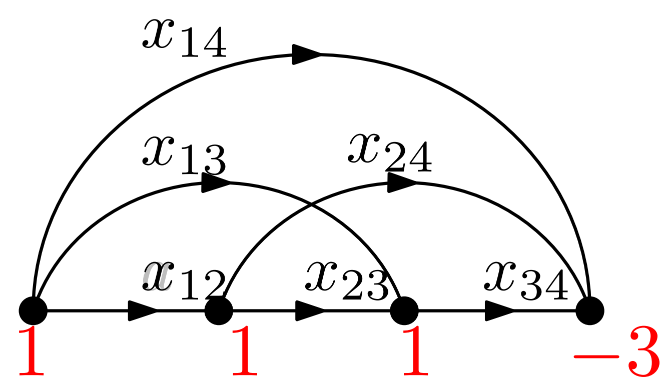

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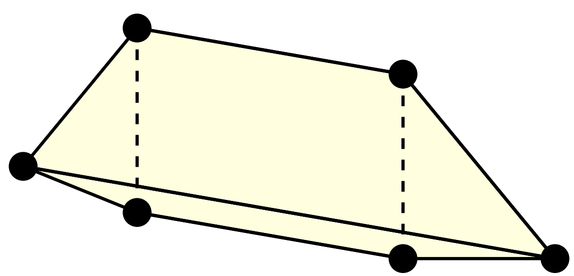

Theorem (M, Morales, Rhoades 2014)

volume equals $\#\,\mathrm{SYT}(n-1, n-2, \ldots, 2, 1) \cdot C_1 C_2 \cdots C_{n-1}$

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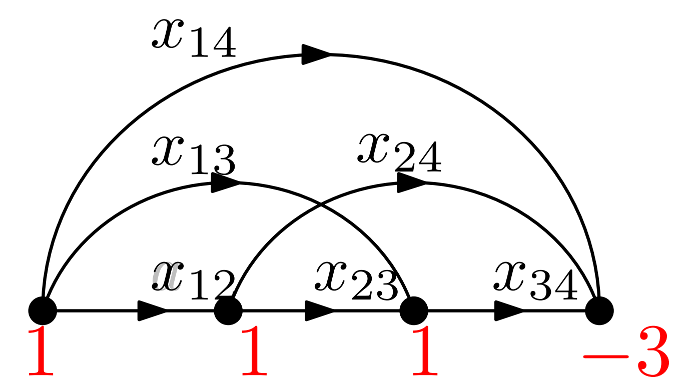

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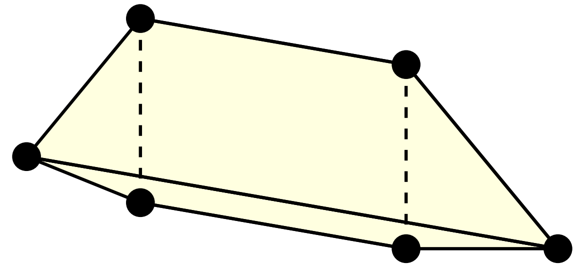
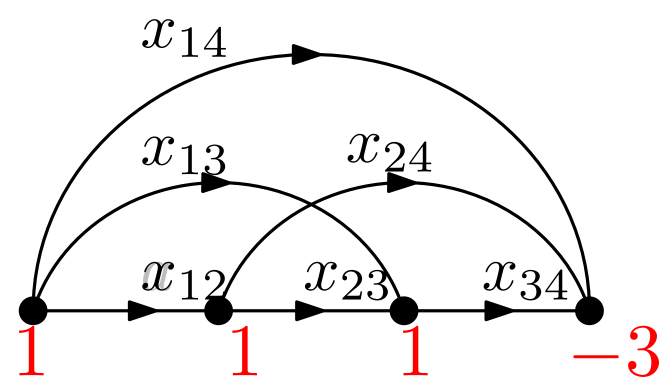


More examples of flow polytopes

Example

G is the complete graph k_{n+1}

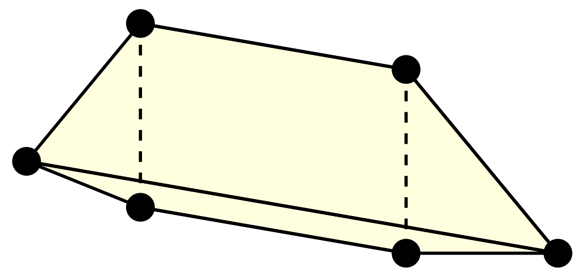
$\mathbf{a} = (1, 1, \ldots, 1, -n)$



Theorem (M, Morales, Rhoades 2014)

volume equals $\#\,\mathrm{SYT}(n-1, n-2, \ldots, 2, 1) \cdot C_1 C_2 \cdots C_{n-1}$

Combinatorial proof?

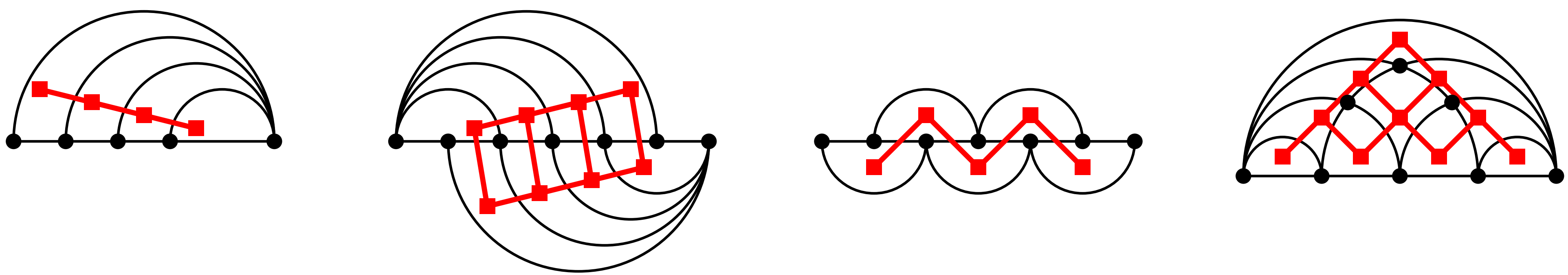


Relation to CRY?

Even more examples of flow polytopes

Theorem (Postnikov 2013)

If G is a planar graph then $\mathcal{F}_G(1,0,\ldots,0,-1)$ is integrally equivalent to an **order polytope** of a certain poset P_G.



Corollary

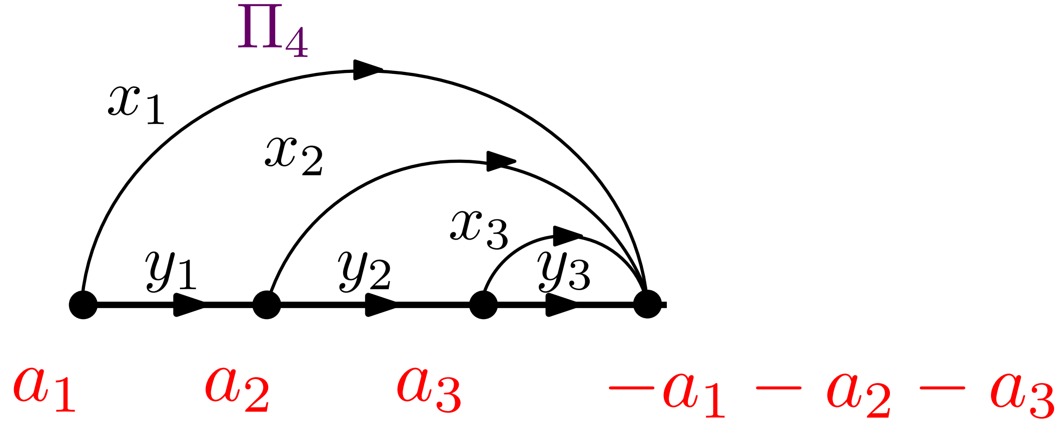
If G is a planar graph then

$$\mathrm{vol}\mathcal{F}_G(1,0,\ldots,0,-1) = \#\text{ linear extensions of } P_G.$$

Examples of flow polytopes

$$\mathcal{F}_G(\mathbf{a}) = \{\text{flows } x(\epsilon) \in \mathbb{R}_{\geq 0},\ \epsilon \in E(G) \mid \text{netflow}(i) = a_i\}$$

Example (Baldoni-Vergne 2008)



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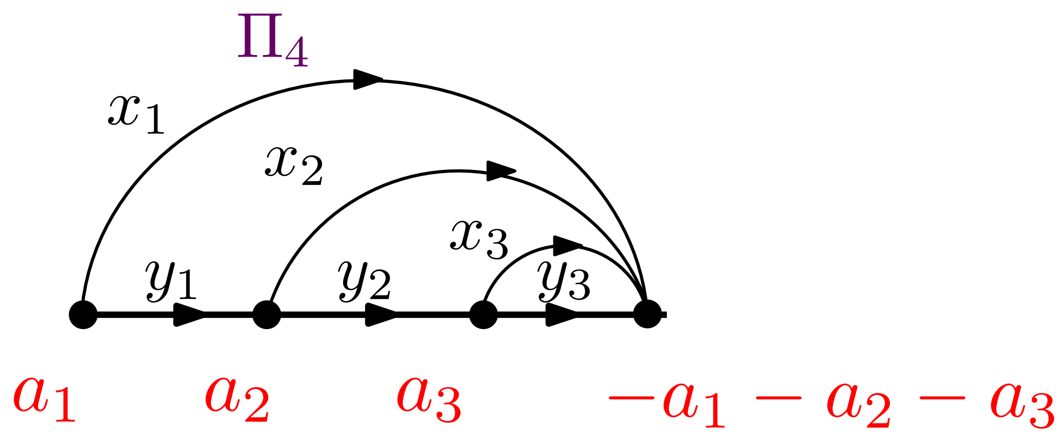

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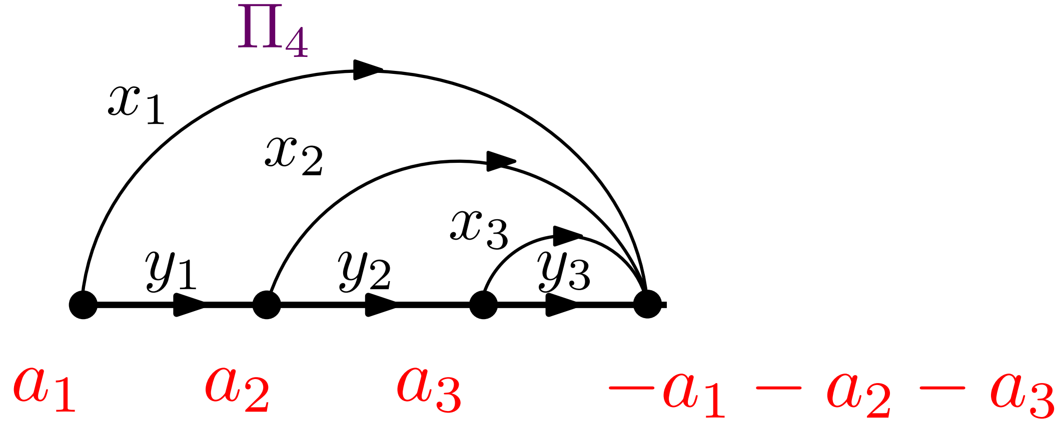

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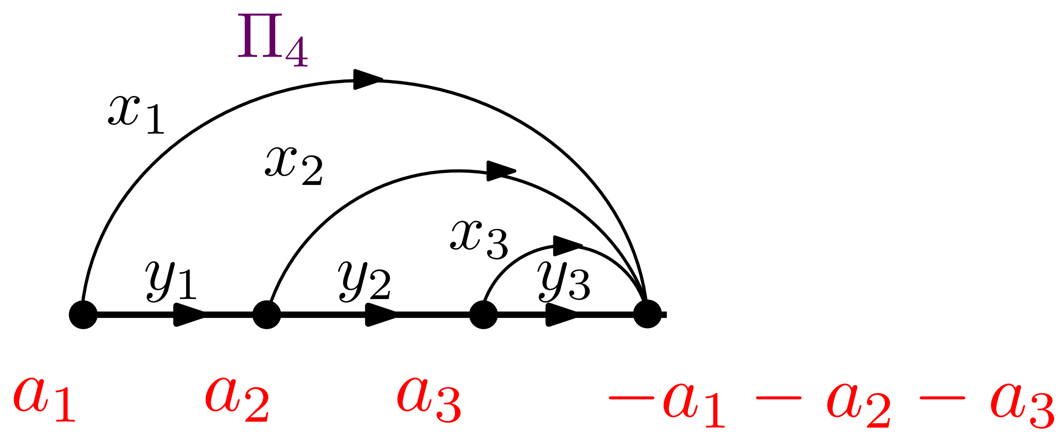

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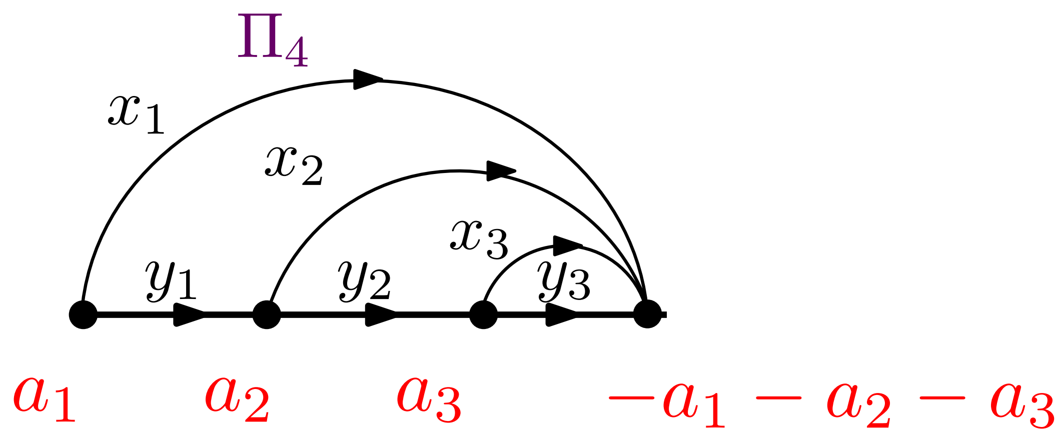

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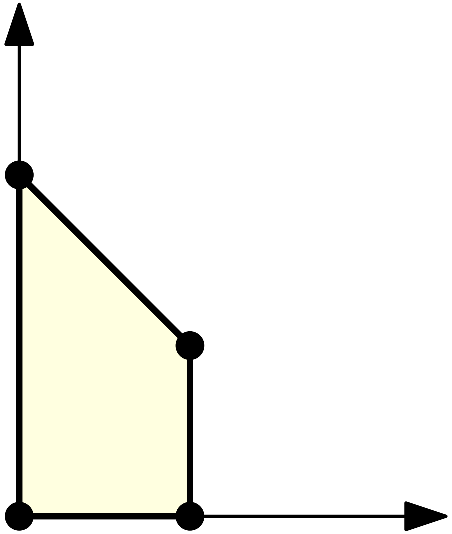
$\mathcal{F}_{\Pi_{n+1}}(\mathbf{a})$ is the Pitman-Stanley polytope

Pitman-Stanley polytope

$\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$

$$\mathrm{PS}_n(\mathbf{a}) = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \;\middle|\; \begin{array}{c} x_1 \leq a_1 \\ x_1 + x_2 \leq a_1 + a_2 \\ \vdots \\ x_1 + \cdots + x_n \leq a_1 + \cdots + a_n \end{array} \right\}$$

Example



$PS_2(1,1)$

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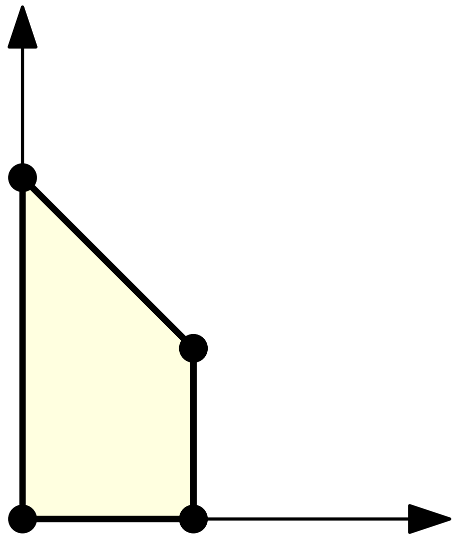

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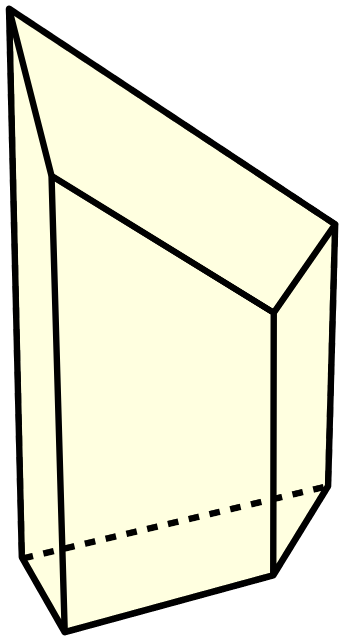

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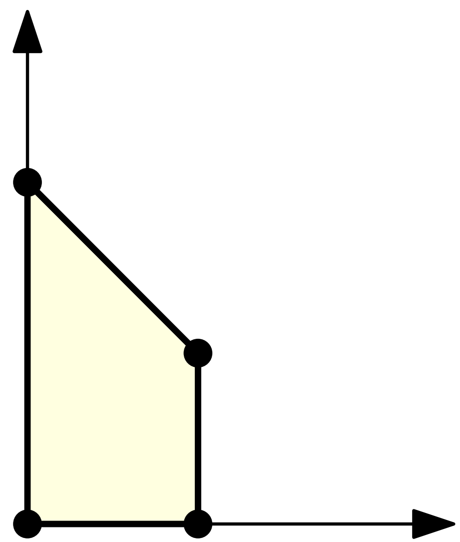

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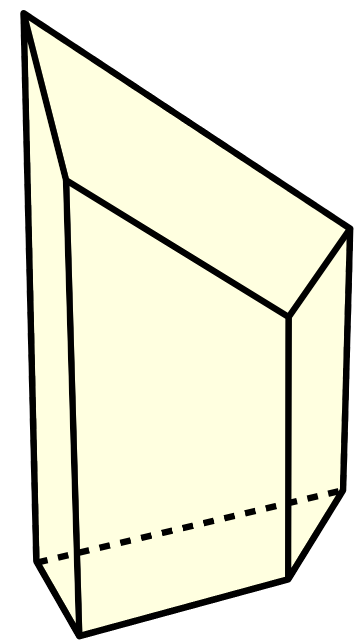
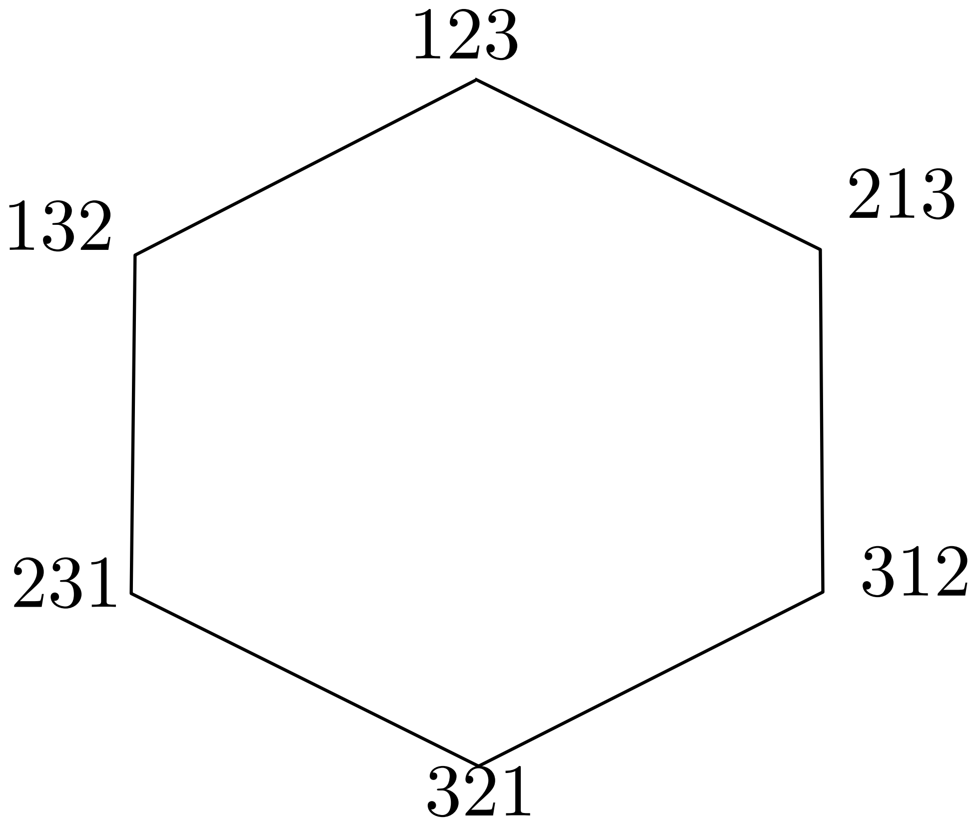


$PS_3(1,1,1)$

- 2^n vertices, n dimensional, is a generalized permutahedron

Generalized permutahedra



Volume of the Pitman-Stanley polytope

Theorem (Pitman-Stanley 01)

$$\operatorname{vol} \mathrm{PS}_n(\mathbf{a}) = \sum_{\mathbf{j} \succeq (1,\ldots,1)} \binom{n}{j_1, \ldots, j_n} a_1^{j_1} \cdots a_n^{j_n}$$

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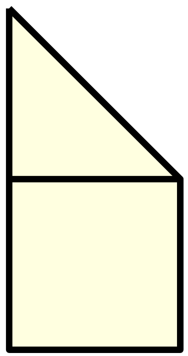

Example

$$\begin{aligned} \mathrm{vol} PS_2(a_1, a_2) &= 2a_1a_2 + a_1^2 \\ &= a_1a_2 + a_2a_1 + a_1^2 \end{aligned}$$

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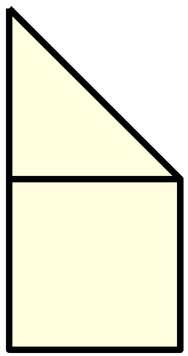

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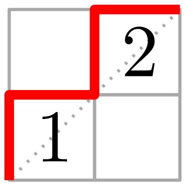

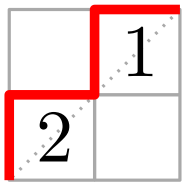

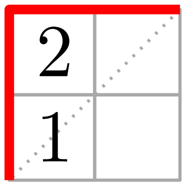

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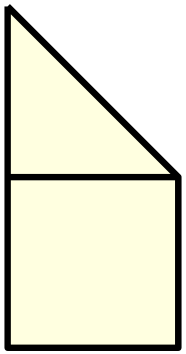

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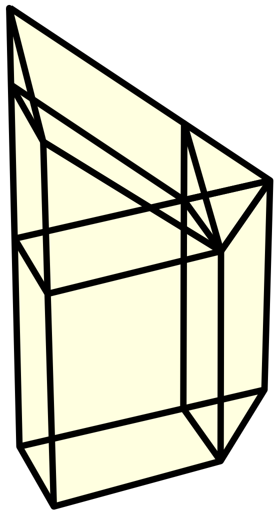

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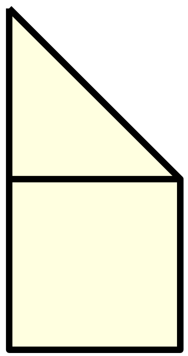

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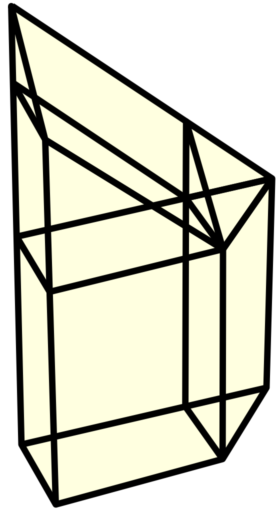

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Proof via a subdivision where each term corresponds to the volume of a cell in subdivision

Lattice points of the Pitman-Stanley polytope

Theorem (Pitman-Stanley, Gessel 01)

$$L_{\mathrm{PS}_n(\mathbf{a})}(t) = \sum_{\mathbf{j} \succeq (1,\ldots,1)} \left(\!\!\binom{a_1 t + 1}{j_1}\!\!\right) \left(\!\!\binom{a_2 t}{j_2}\!\!\right) \cdots \left(\!\!\binom{a_n t}{j_n}\!\!\right)$$

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$\left(\!\binom{3}{2}\!\right) = 6$, counting $\{1,1\}, \{1,2\}, \{1,3\}, \{2,2\}, \{2,3\}, \{3,3\}$

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$\left(\!\binom{m}{n}\!\right) = \binom{m+n-1}{n}$

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Corollary

$$L_{\mathrm{PS}_n(\mathbf{a})}(t) \in \mathbb{N}[t]$$

Summary

$$\mathcal{F}_G(\mathbf{a}) = \{\text{flows } x(\epsilon) \in \mathbb{R}_{\geq 0},\ \epsilon \in E(G) \mid \text{netflow}(i) = a_i\}$$

Examples

- $\mathcal{F}_{k_{n+1}}(\mathbf{a})$: CRY polytope ($\mathbf{a} = (1, 0, \ldots, 0, -1)$), Tesler polytope ($\mathbf{a} = (1, 1, \ldots, 1, -n)$);

  volumes divisible by $C_1 \cdots C_{n-2}$

- $\mathcal{F}_{\Pi_{n+1}}(\mathbf{a})$: **Pitman-Stanley polytope**, explicit volume and lattice point formulas related to parking functions.

Question

- Is there a formula for volume and lattice points of $\mathcal{F}_G(\mathbf{a})$?

Lidskii volume formula

Theorem (Baldoni-Vergne 08, Postnikov-Stanley - unpublished)
G m edges, $n+1$ vertices, $a_i \geq 0$

$$\mathrm{vol}\mathcal{F}_G(a_1,\ldots,a_n) = \sum_{\mathbf{j}\succeq\mathbf{o}} \binom{m-n}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n} \times K_G(j_1-o_1,\ldots,j_n-o_n,0)$$

where $\mathbf{o}=(o_1,\ldots,o_n)$, $o_v = outdeg(v)-1$ and $|\mathbf{j}|=m-n$.

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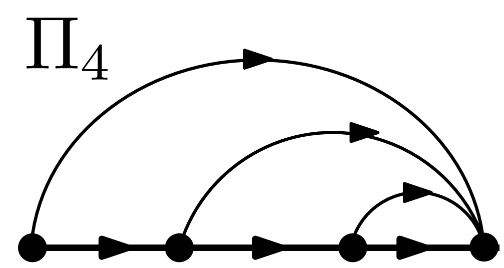

$$\mathrm{vol}\mathcal{F}_{\Pi_{n+1}}(\mathbf{a}) = \sum_{\mathbf{j}\succeq(1,\ldots,1)} \binom{n}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n}\cdot 1$$

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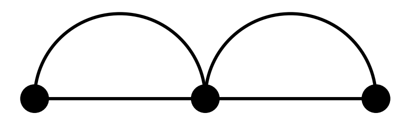
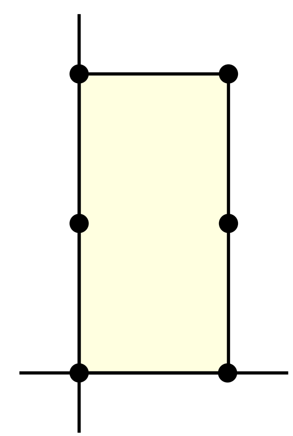
Example



$$\mathrm{vol}\mathcal{F}_G(\mathbf{1}) = \binom{2}{1,1} K_G(1-1,1-1,0) + \binom{2}{2,0} K_G(2-1,0-1,0)$$
$$= 2\cdot 1 + 1\cdot 2 = 4.$$



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Corollary:

$$\mathrm{vol}\mathcal{F}_G(1,0,\ldots,0,-1) = 1\cdot K_G(m-n-o_1,-o_2,\ldots,-o_n,0).$$

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Example: (CRY polytope)

$$\mathrm{vol}\mathcal{F}_{k_{n+1}}(1,0,\ldots,0,-1) = K_{k_{n+1}}(\tbinom{n-1}{2},-n+2,\ldots,-2,-1,0)$$

Lidskii lattice point formula

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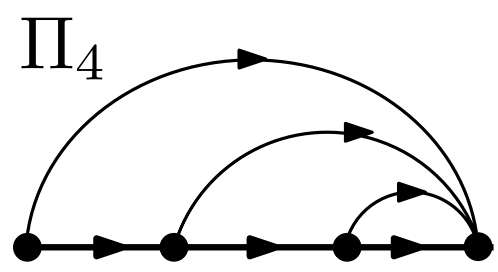
Pitman-Stanley polytope:



$$\mathcal{F}_{\Pi_{n+1}}(\mathbf{a}) = \sum_{\mathbf{j}\succeq(1,\ldots,1)} \left(\!\!\binom{a_1+1}{j_1}\!\!\right)\left(\!\!\binom{a_2}{j_2}\!\!\right)\cdots\left(\!\!\binom{a_n}{j_n}\!\!\right)$$

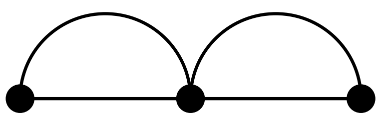
Lidskii lattice point formula

Theorem (Baldoni-Vergne 08, Postnikov-Stanley – unpublished)
G m edges, $n+1$ vertices, $a_i \geq 0$

$$K_G(a_1,\ldots,a_n) = \sum_{\mathbf{j}\succeq\mathbf{o}} \left(\!\!\binom{a_1-i_1}{j_1}\!\!\right)\cdots\left(\!\!\binom{a_n-i_n}{j_n}\!\!\right) \times K_G(j_1-o_1,\ldots,j_n-o_n,0)$$

where $|\mathbf{j}| = m-n$, $o_v = outdeg(v)-1$, $i_v = indeg(v)-1$

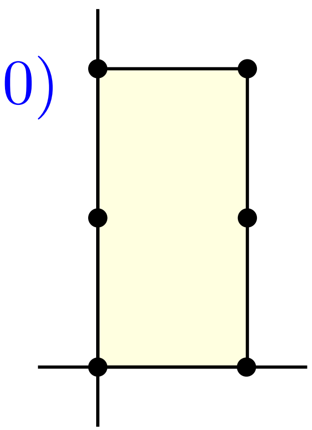
Example



$K_G(1,1,-2) =$

$$= \left(\!\!\binom{2}{1}\!\!\right)\left(\!\!\binom{0}{1}\!\!\right) K_G(0,0,0) + \left(\!\!\binom{2}{2}\!\!\right)\left(\!\!\binom{0}{0}\!\!\right) K_G(1,-1,0)$$

$$= 0 + 3\cdot 2 = 6.$$



About the proofs

$$K_G(a_1,\ldots,a_n) = \sum_{\mathbf{j}\succeq\mathbf{o}} \left(\!\!\binom{a_1-i_1}{j_1}\!\!\right) \cdots \left(\!\!\binom{a_n-i_n}{j_n}\!\!\right) \times K_G(j_1-o_1,\ldots,j_n-o_n,0)$$

- proof by Baldoni and Vergne uses residues
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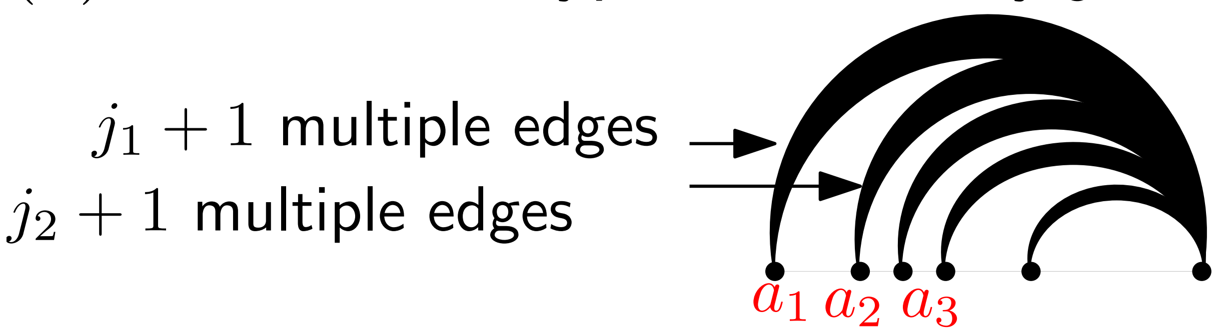
$$\times K_G(j_1 - o_1, \ldots, j_n - o_n, 0)$$

- proof by Baldoni and Vergne uses residues
- proof by Postnikov-Stanley uses the Elliott-MacMahon algorithm
- new proof (M-Morales) by polytope subdivision

Subdivision proof of Lidskii formulas

$$\mathrm{vol}\mathcal{F}_G(a_1,\ldots,a_n) = \sum_{\mathbf{j}\succeq\mathbf{o}} \binom{m-n}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n} \times K_G(j_1-o_1,\ldots,j_n-o_n,0)$$

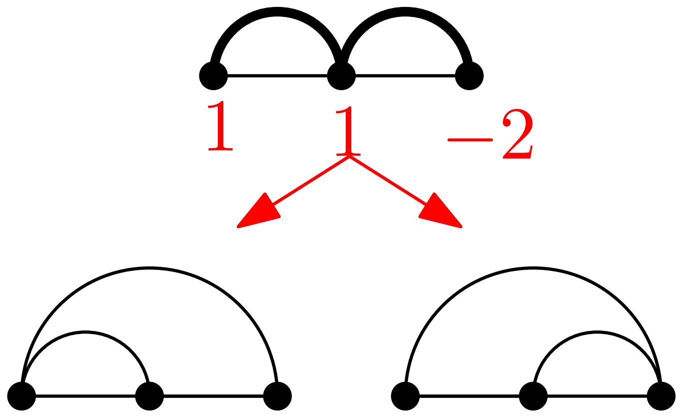
Subdivide $\mathcal{F}_G(\mathbf{a})$ into **cells** of types indexed by $\mathbf{j}$.



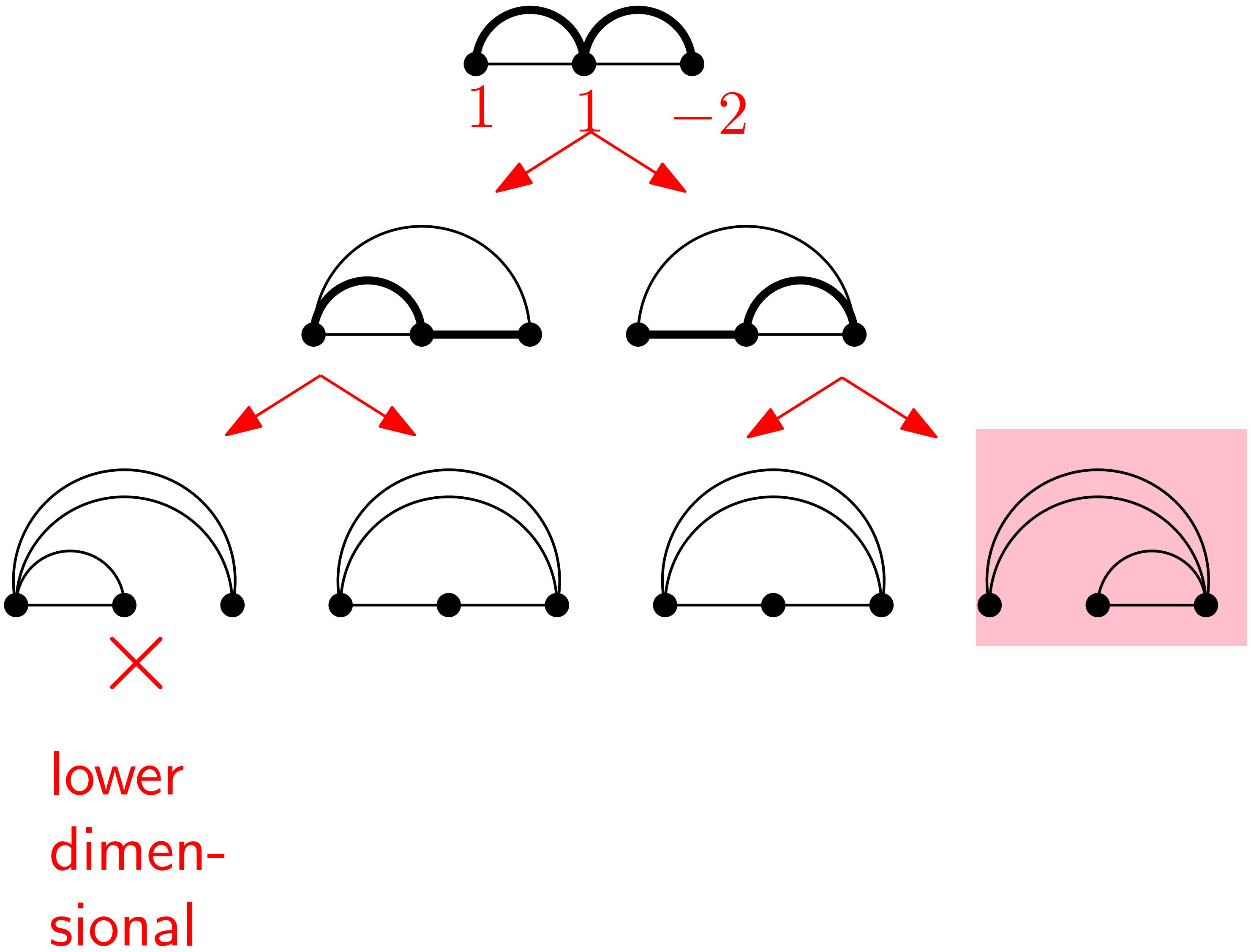
volume of each type $\mathbf{j}$ cell : $\binom{m-n}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n}$

times type $\mathbf{j}$ cell appears: $K_G(j_1-o_1,\ldots,j_n-o_n,0)$

Example subdivision



Example subdivision



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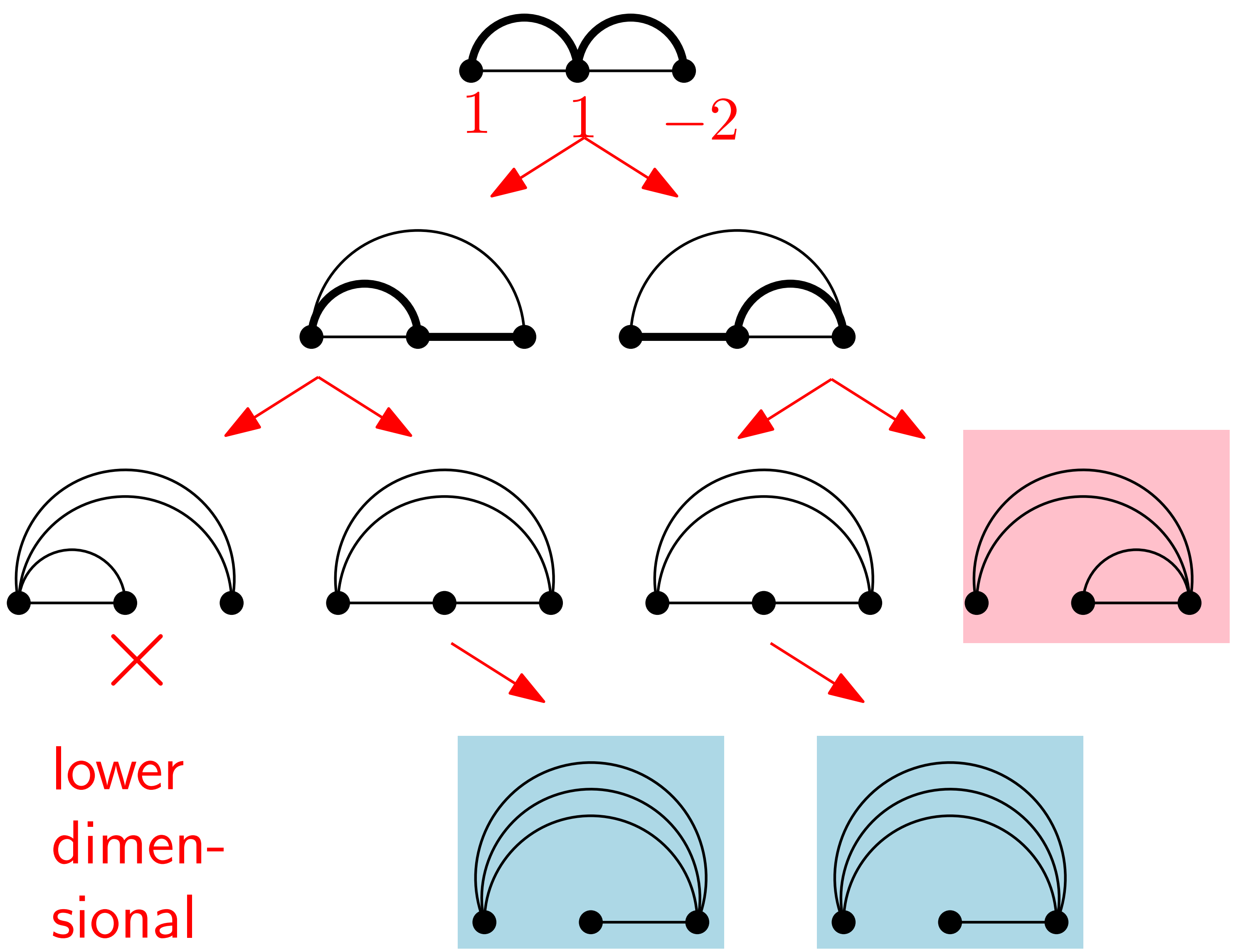

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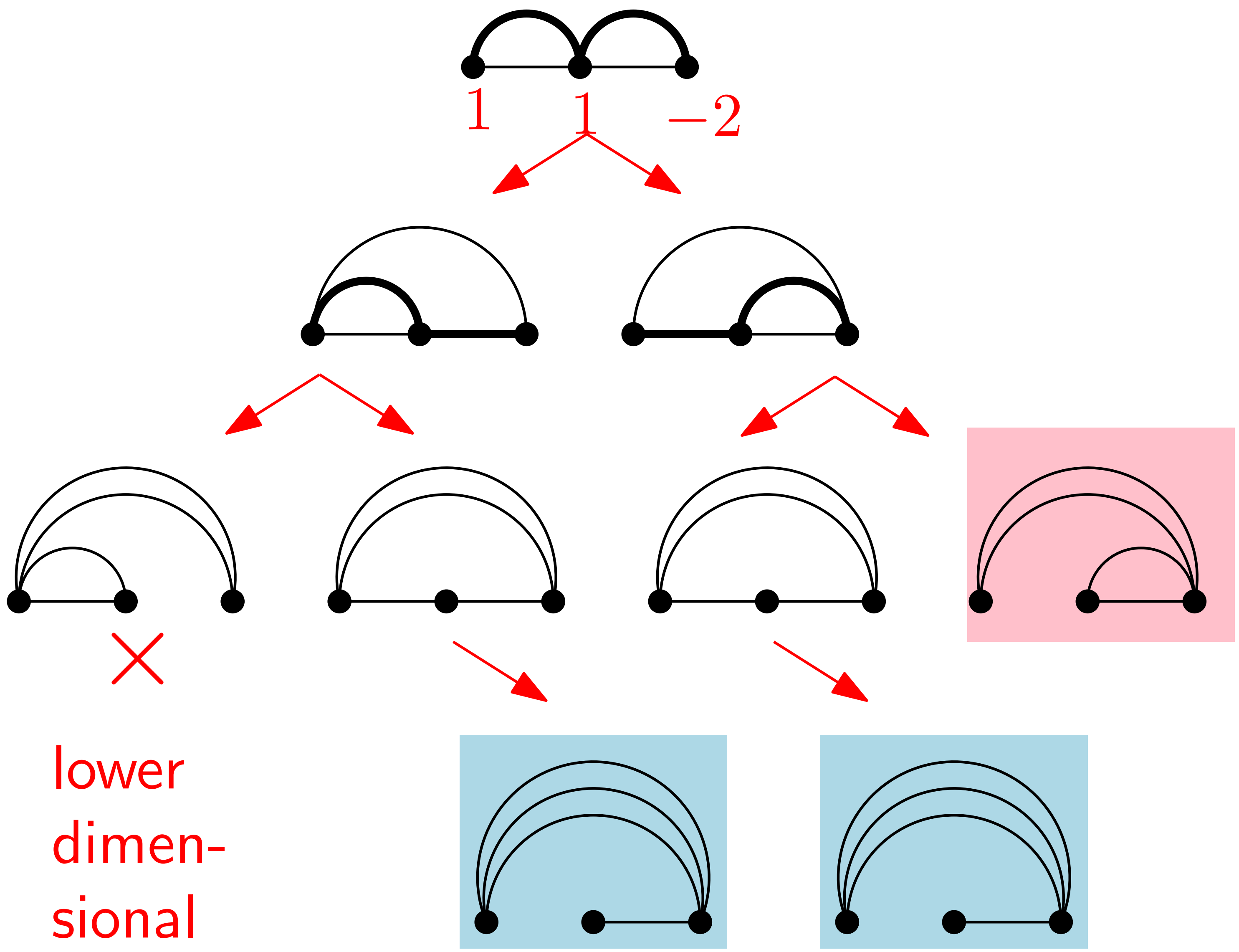

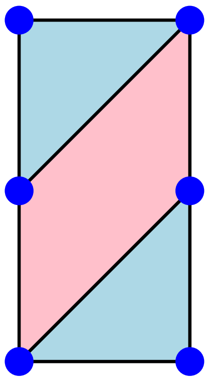

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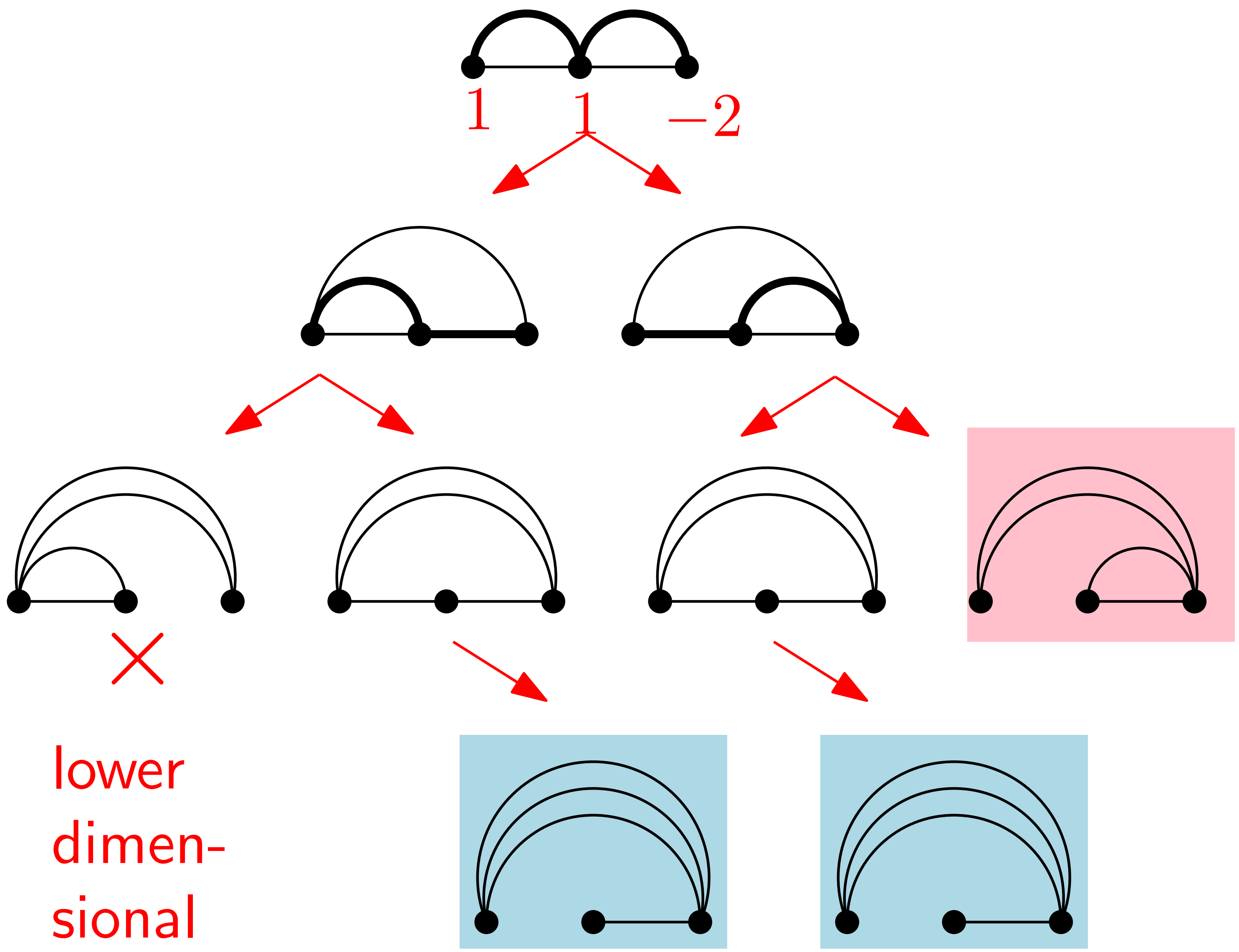

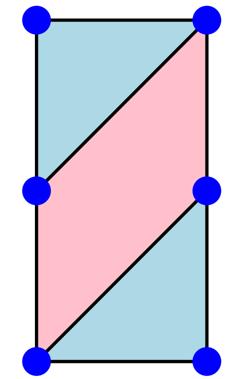

volume:
$2 \cdot 1 + 1 \cdot 2 = 4.$

lattice points:
$0 + 3 \cdot 2 = 6.$

Triangulation in the case $\mathcal{F}_G(1, 0, \ldots, 0, -1)$

Special case (Stanley-Postnikov)

$$\mathrm{vol}\mathcal{F}_G(1, 0, \ldots, 0, -1) = 1 \cdot K_G(m - n - o_1, -o_2, \ldots, -o_n, 0).$$

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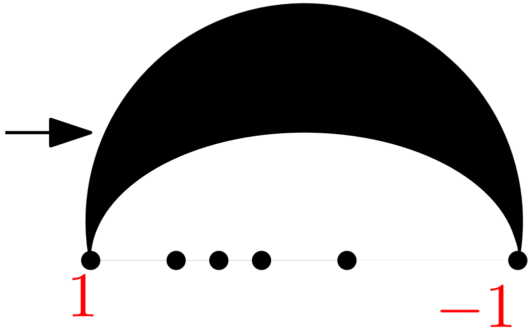
Final outcomes of subdivision all of the form

$m - n + 1$ multiple edges



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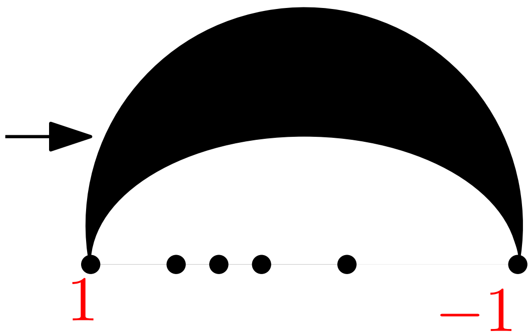

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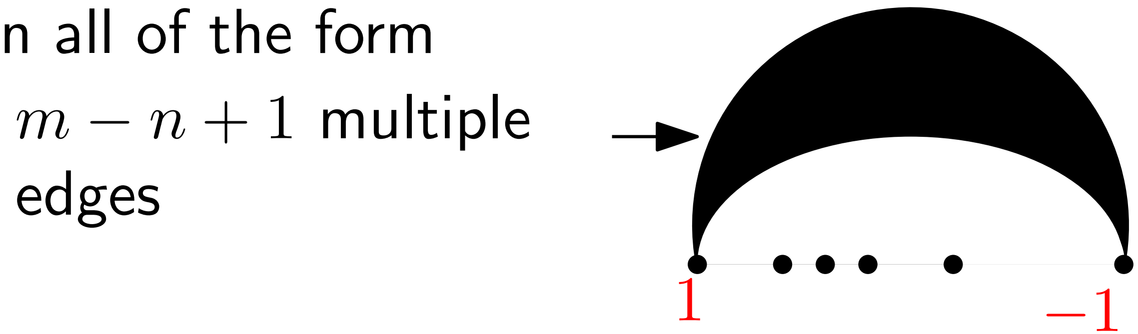

- the associated polytope is a simplex with volume 1

- They proved the number of times we obtain this outcome in the subdivision tree is $K_G(m - n - o_1, -o_2, \ldots, -o_n, 0)$

Flow polytopes and...

Flow polytopes and...

- Kostant partition functions

Flow polytopes and...

- Kostant partition functions ✓

Flow polytopes and...

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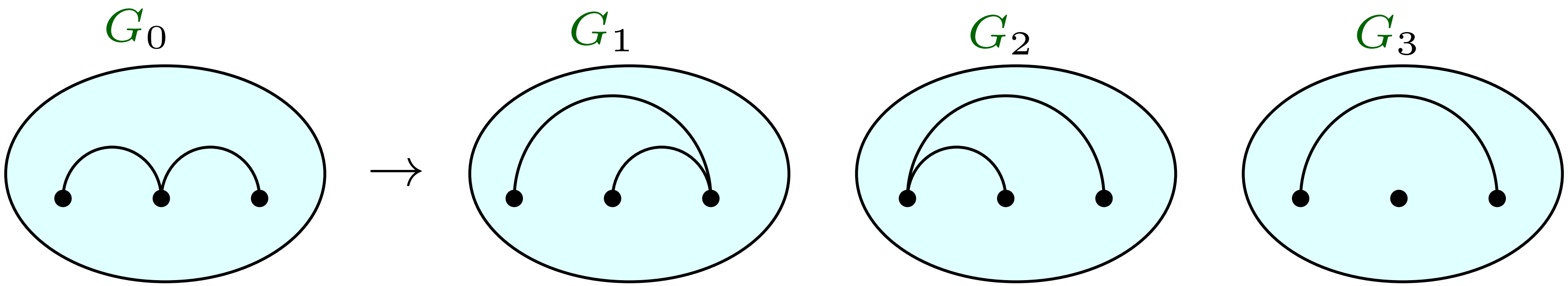
- Grothendieck polynomials

Flow polytopes and...

- Kostant partition functions ✓
- Grothendieck polynomials
- space of diagonal harmonics

Encoding triangulations of $\mathcal{F}(G) := \mathcal{F}_G(1, 0, \ldots, 0, -1)$



Reduction Lemma.

$$\mathcal{F}(G_0) = \mathcal{F}(G_1) \cup \mathcal{F}(G_2),$$
$$\mathcal{F}(G_1) \cap \mathcal{F}(G_2) = \mathcal{F}(G_3).$$

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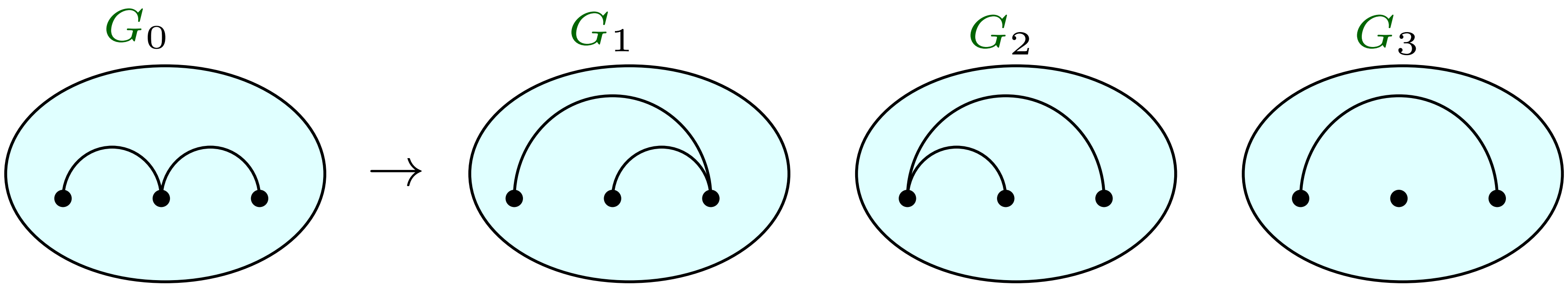

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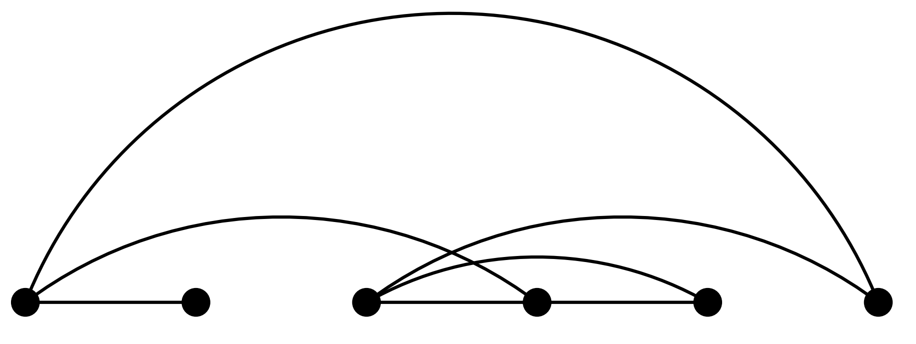
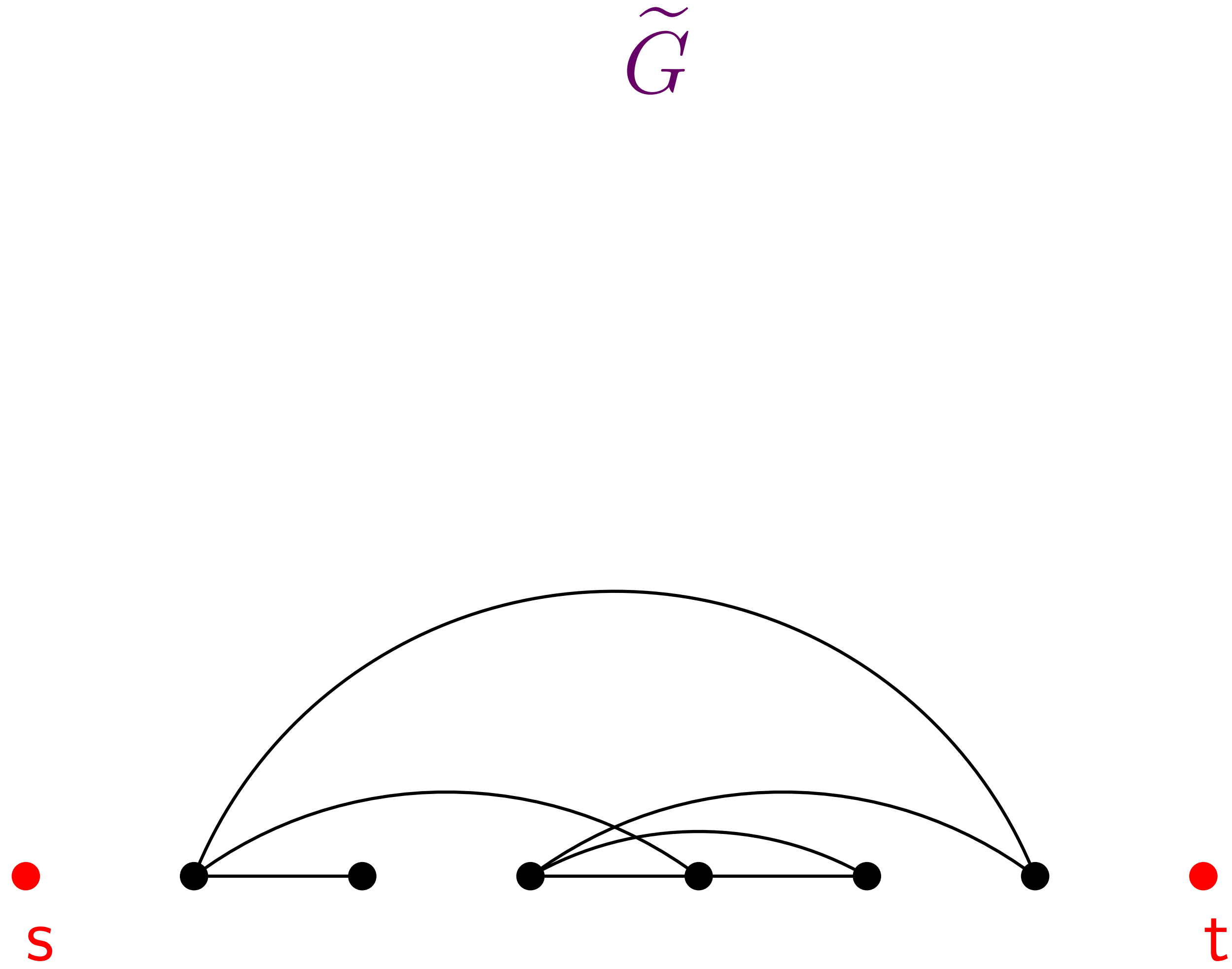
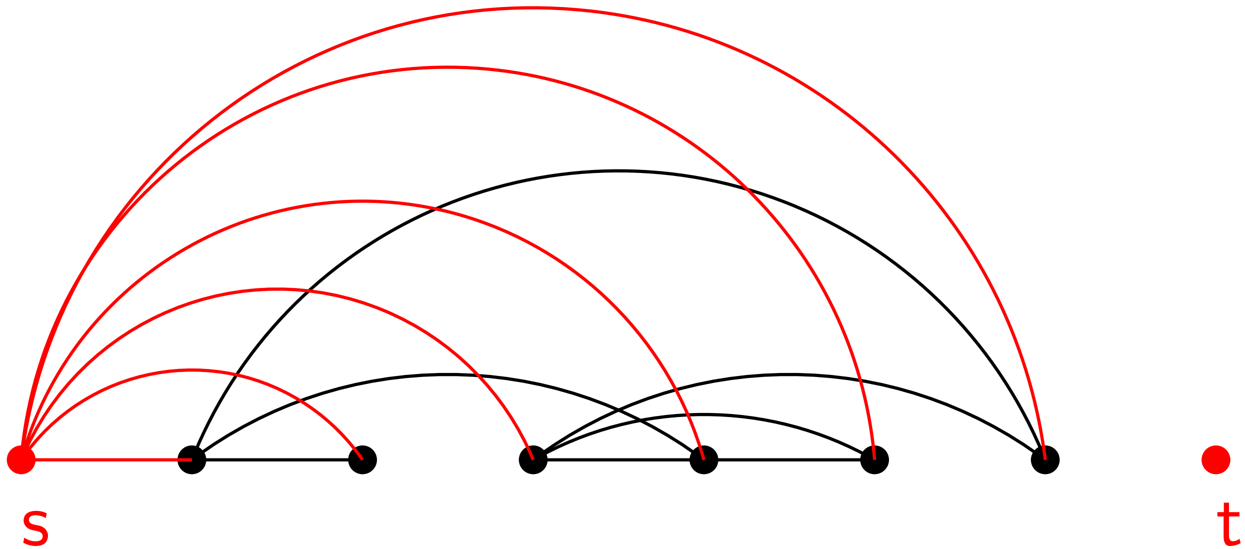
Some of the polytopes above may be empty.

$\widetilde{G}$

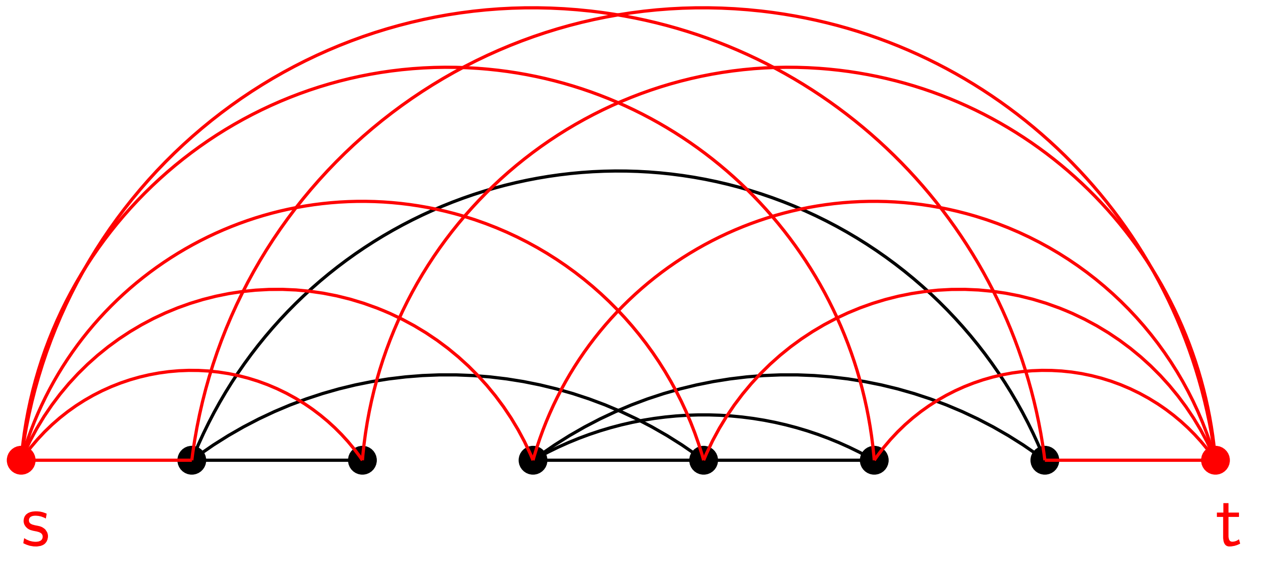



$\widetilde{G}$
s
t

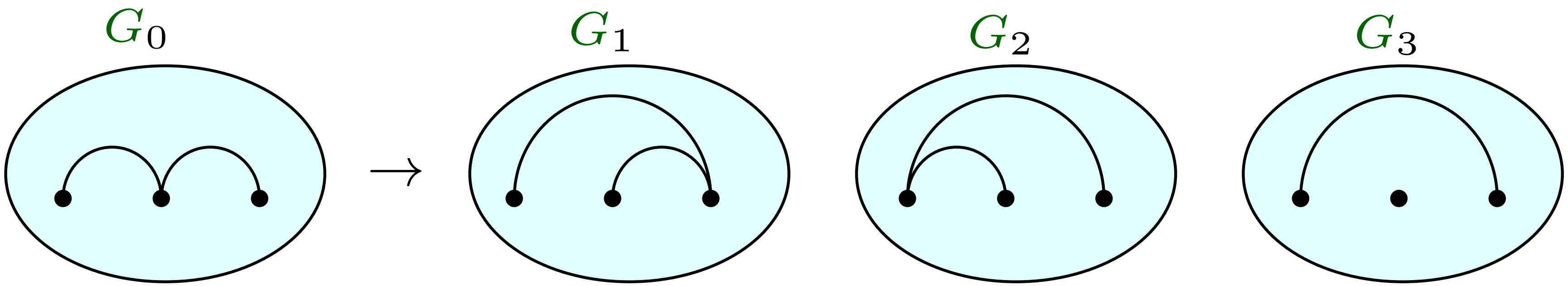
$\widetilde{G}$



$\widetilde{G}$



Encoding triangulations of $\mathcal{F}(\widetilde{G})$



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$$\mathcal{F}(\widetilde{G}_0) = \mathcal{F}(\widetilde{G}_1) \cup \mathcal{F}(\widetilde{G}_2),$$

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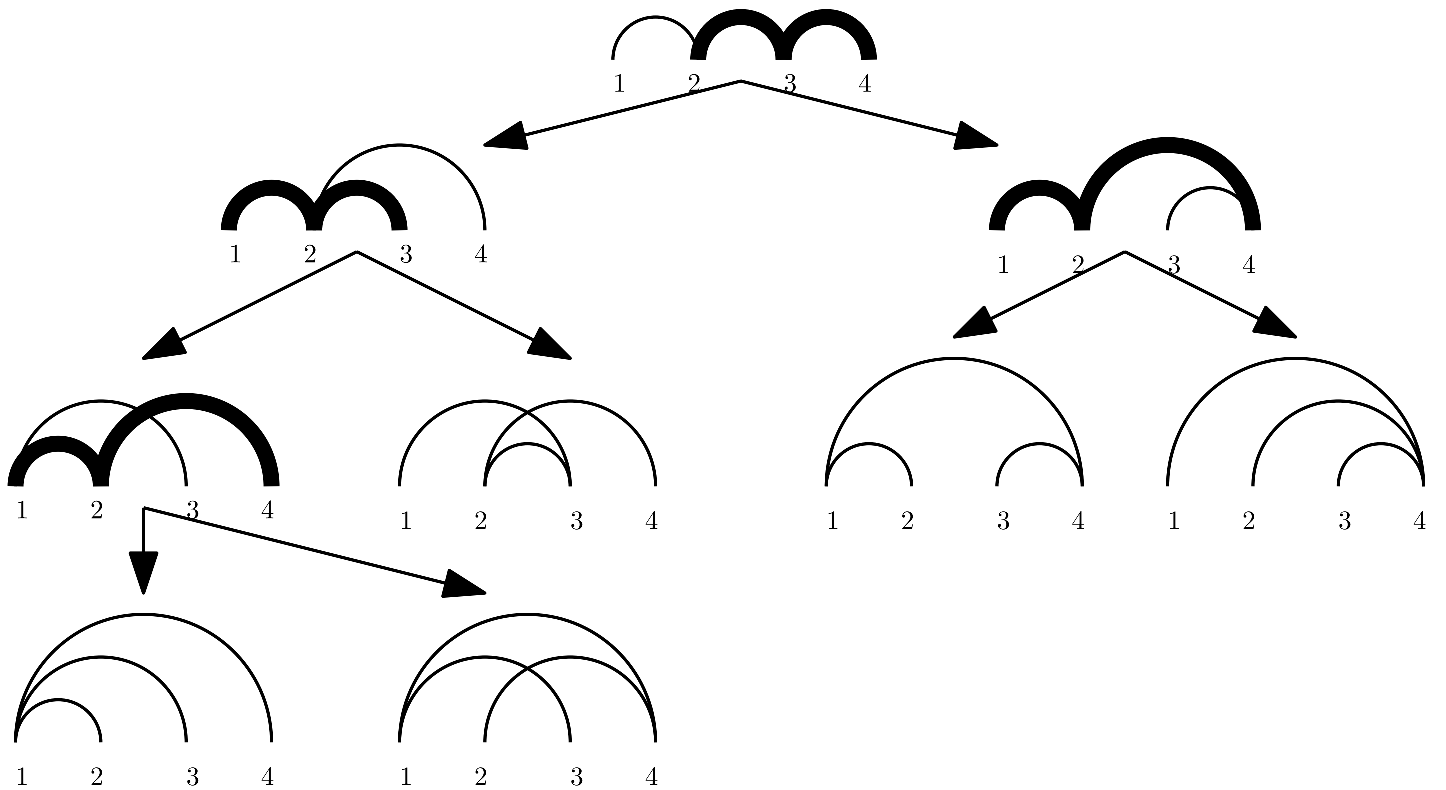
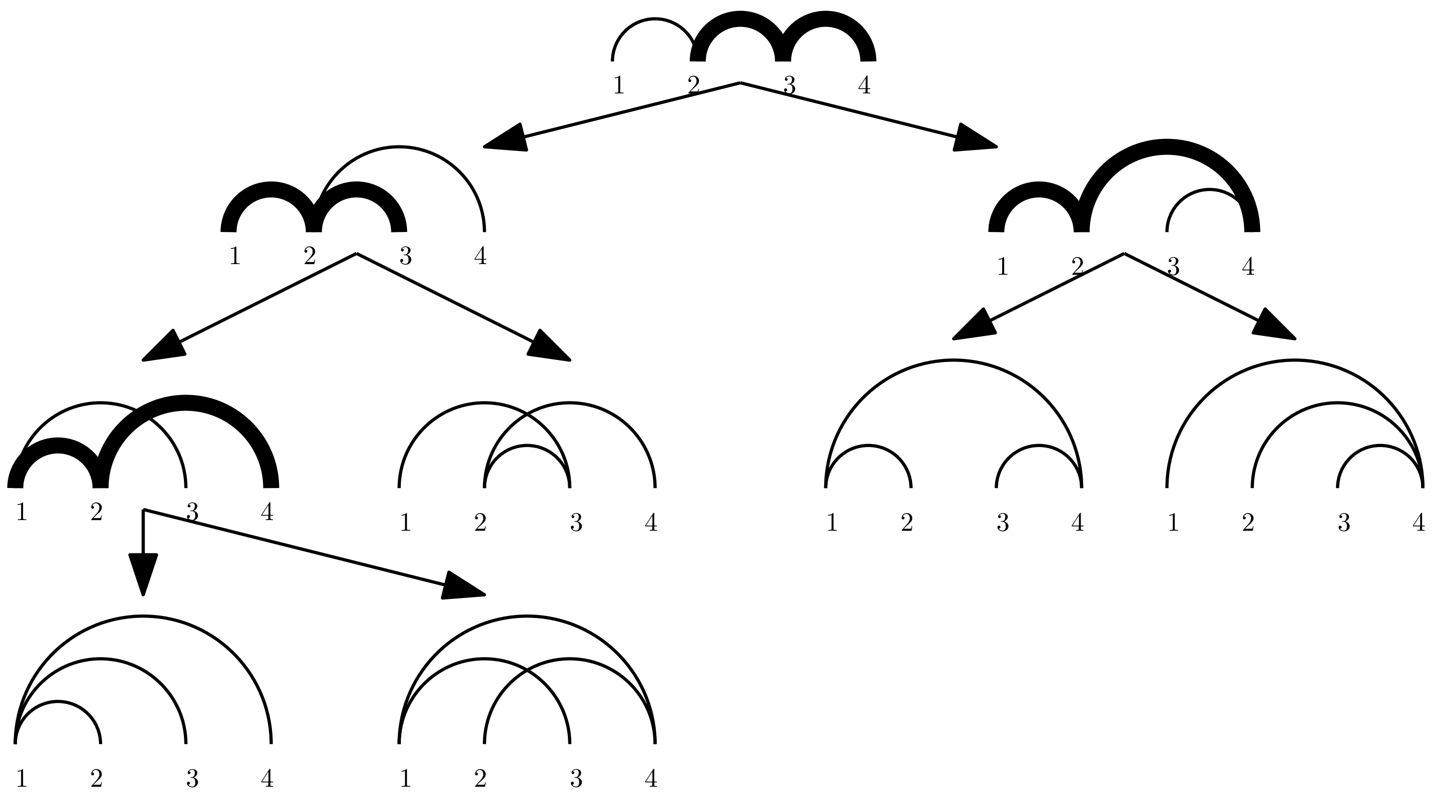
where $\mathcal{F}(\widetilde{G}_0), \mathcal{F}(\widetilde{G}_1), \mathcal{F}(\widetilde{G}_2)$, are of the same dimension and $\mathcal{F}(\widetilde{G}_3)$ is one dimension less.

Reduction tree $\mathcal{T}(G)$



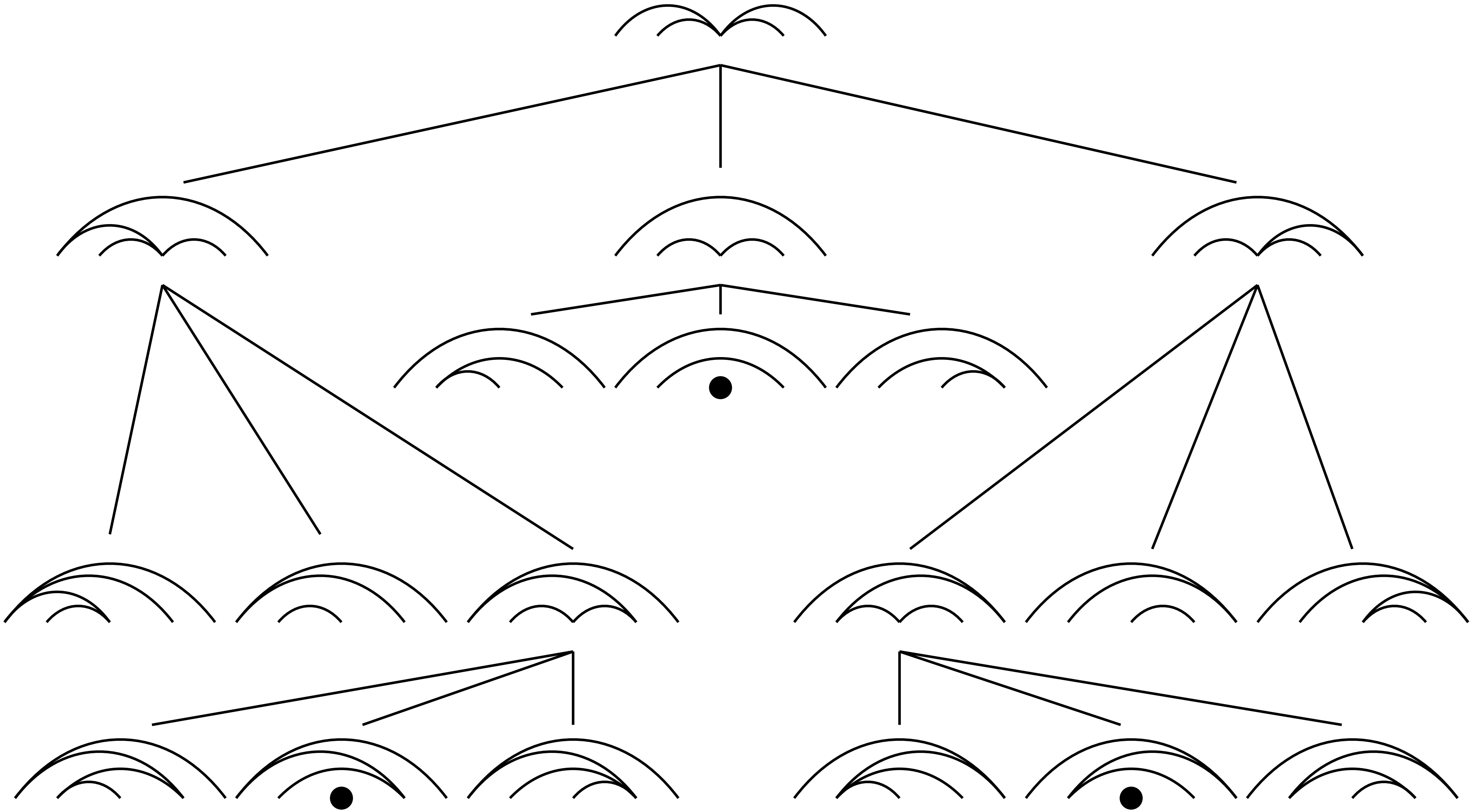
A reduction tree of $G = ([4], \{(1,2),(2,3),(3,4)\})$ with five leaves. The edges on which the reductions are performed are in bold.

Reduction tree $\mathcal{T}(G)$



If the leaves are labeled by graphs $H_1, \ldots, H_k$ then the flow polytopes $\mathcal{F}(\widetilde{H}_1), \ldots, \mathcal{F}(\widetilde{H}_k)$ are simplices.

Canonical reduction tree



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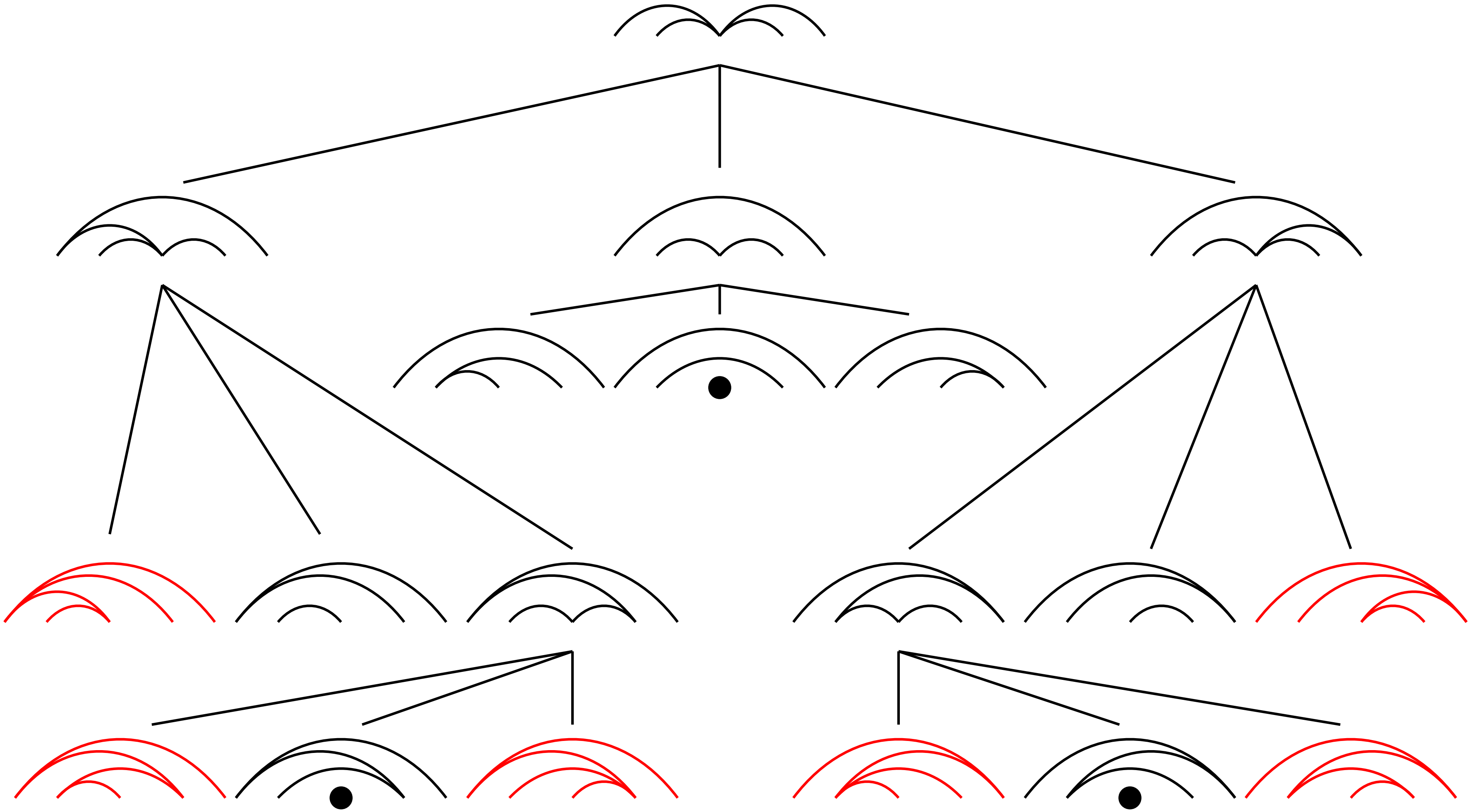
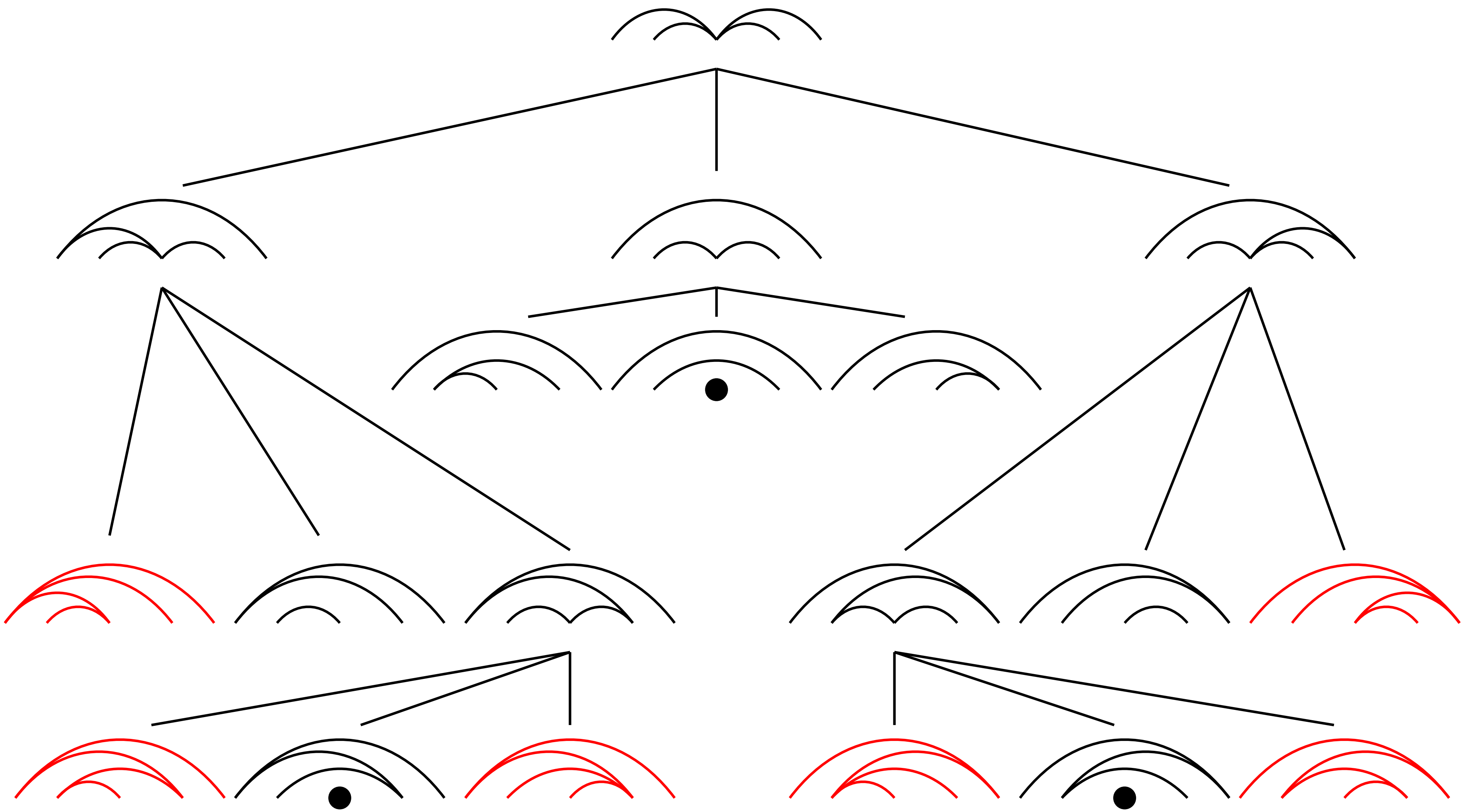
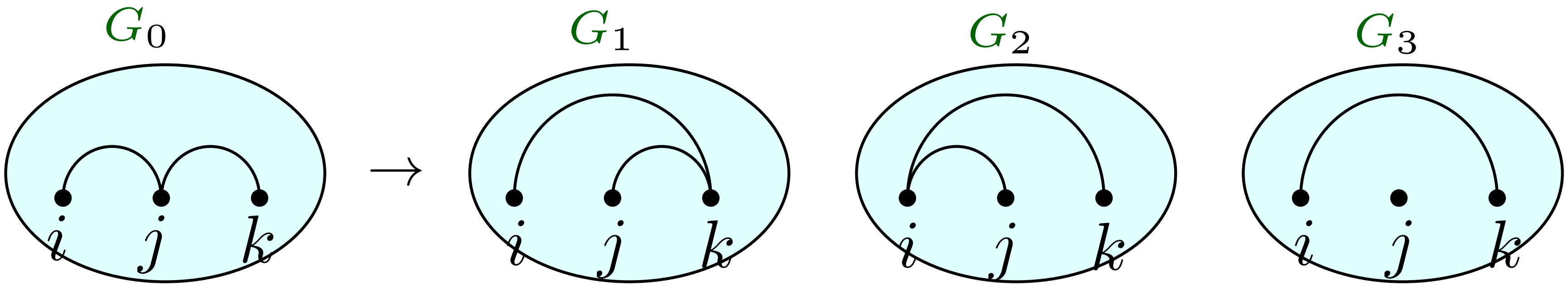


Canonical reduction tree



Theorem (M, 2009) The full dimensional leaves of the canonical reduction tree are the noncrossing alternating spanning trees of the directed transitive closure of the noncrossing tree at the root.

Encoding triangulations with reduced forms



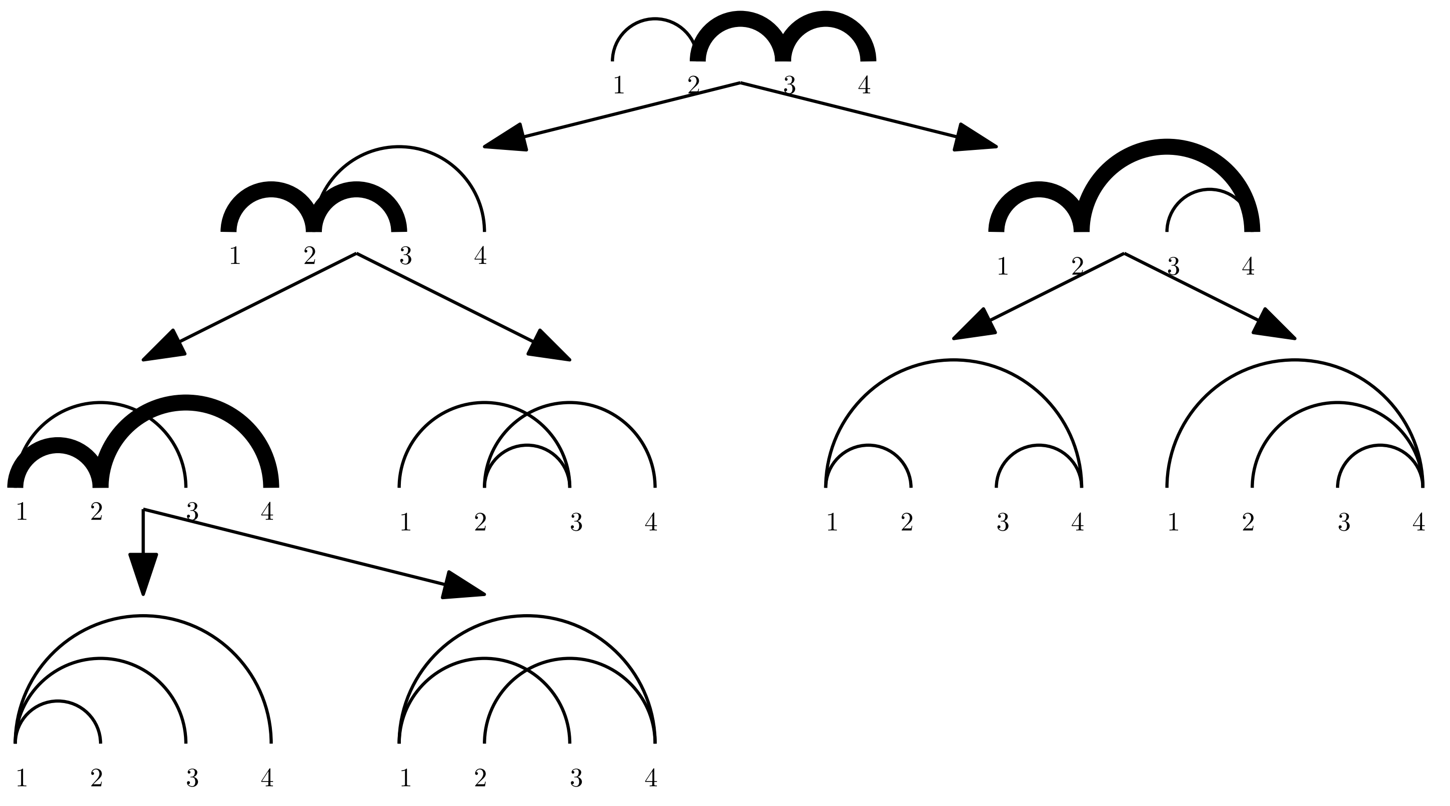
$$x_{ij}x_{jk} \rightarrow x_{jk}x_{ik} + x_{ik}x_{ij} + \beta x_{ik}$$

The subdivision algebra

Generated by x_{ij}, $1 \leq i < j \leq n$, over $\mathbb{Z}[\beta]$ subject to relations

- $x_{ij}x_{kl} = x_{kl}x_{ij}$, for all i, j, k, l
- $x_{ij}x_{jk} = x_{jk}x_{ik} + x_{ik}x_{ij} + \beta x_{ik}$

Reduced form



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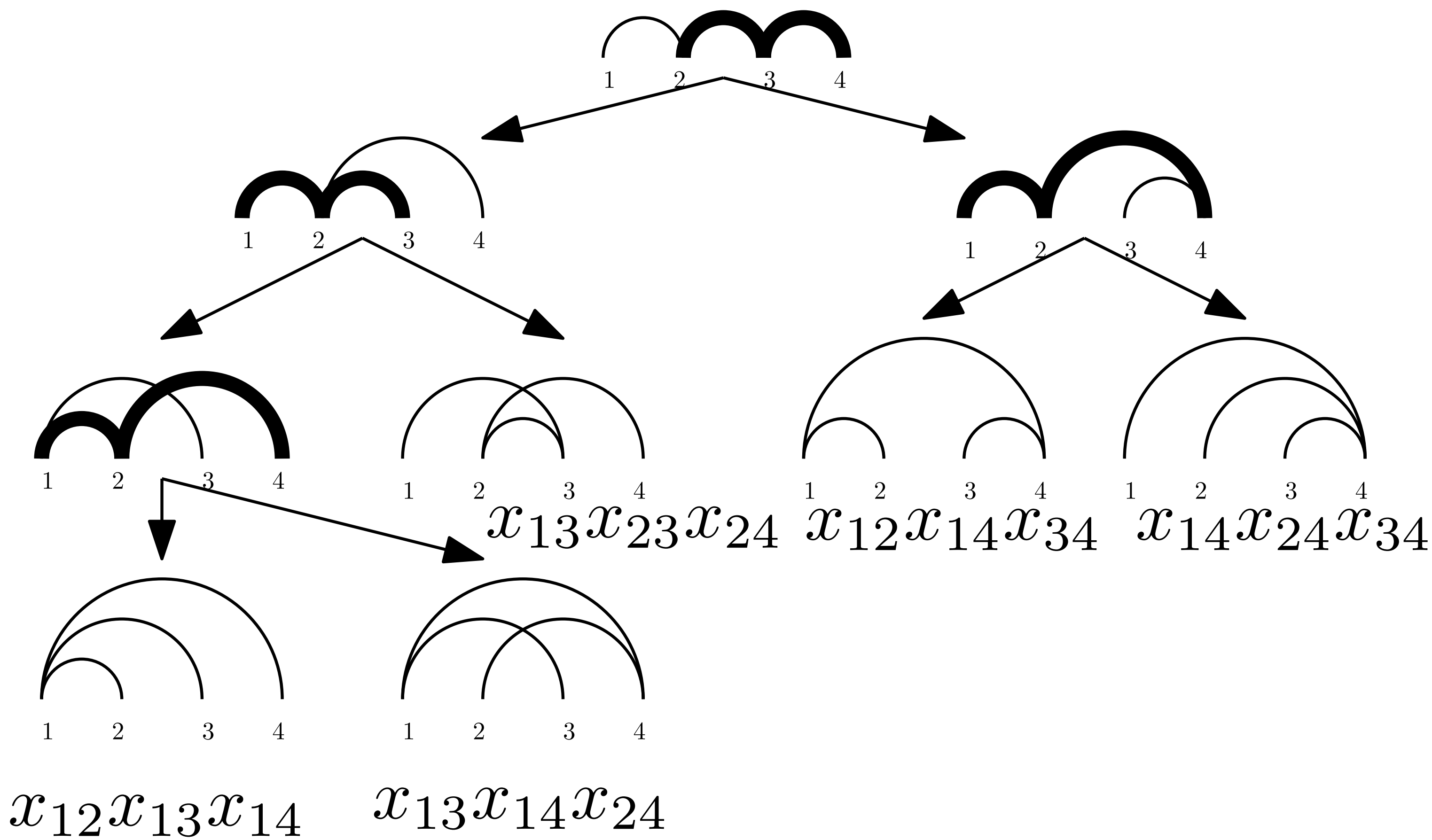
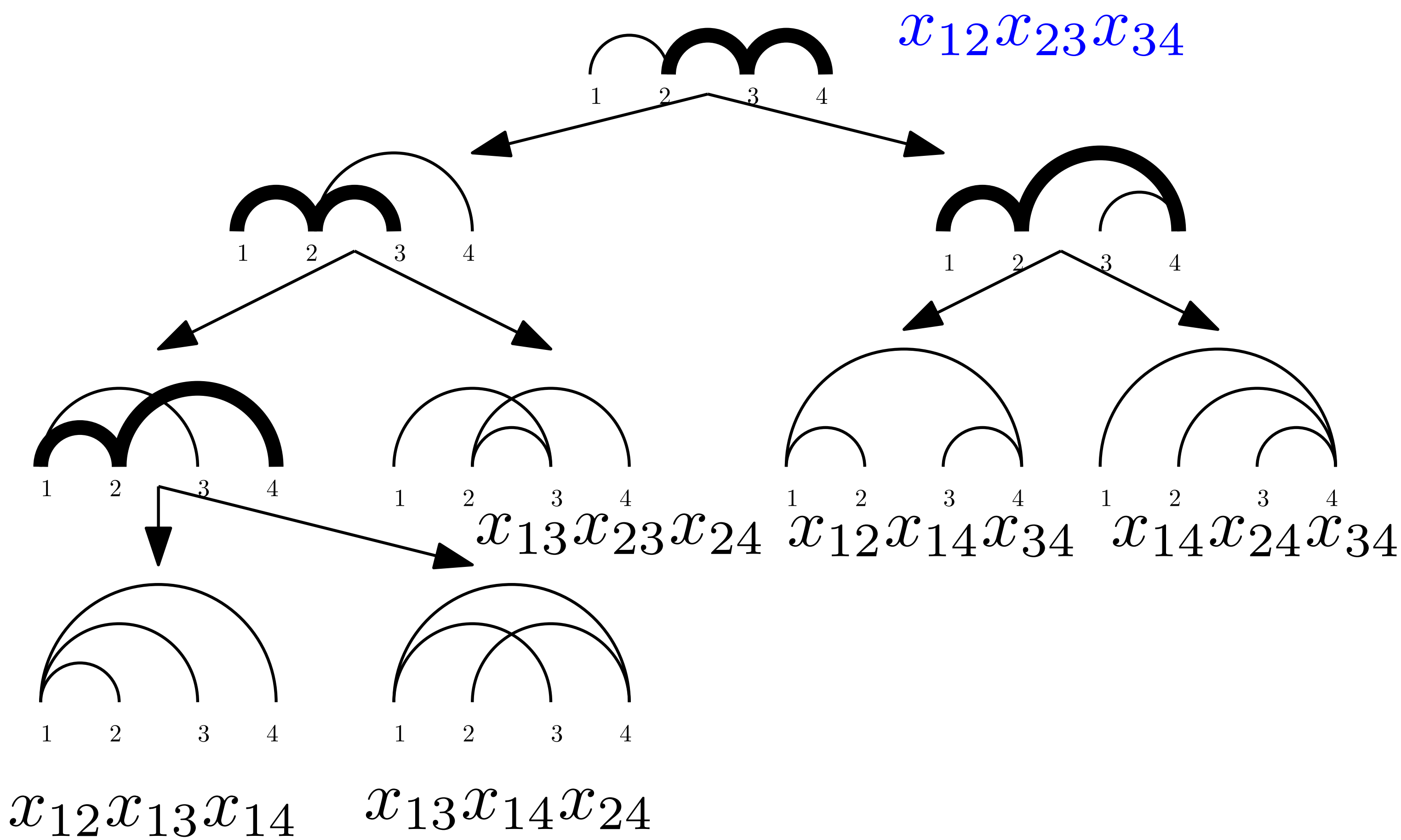


Reduced form



$$x_{12}x_{13}x_{14} + x_{13}x_{14}x_{24} + x_{13}x_{23}x_{24} + x_{12}x_{14}x_{34} + x_{14}x_{24}x_{34}$$

$(\beta = 0)$

Reduced form

Denote by $Q_G^{\beta}(\mathbf{x})$ a reduced form of the monomial $\prod_{(i,j)\in E(G)} x_{ij}$.

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$Q_G^\beta(\mathbf{1})$ is independent of the reductions performed.

Uniqueness of reduced forms

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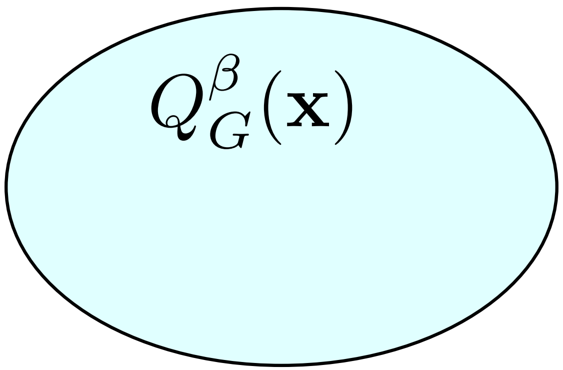

Uniqueness of reduced forms

$Q_G^{\beta}(\mathbf{x})$

not unique

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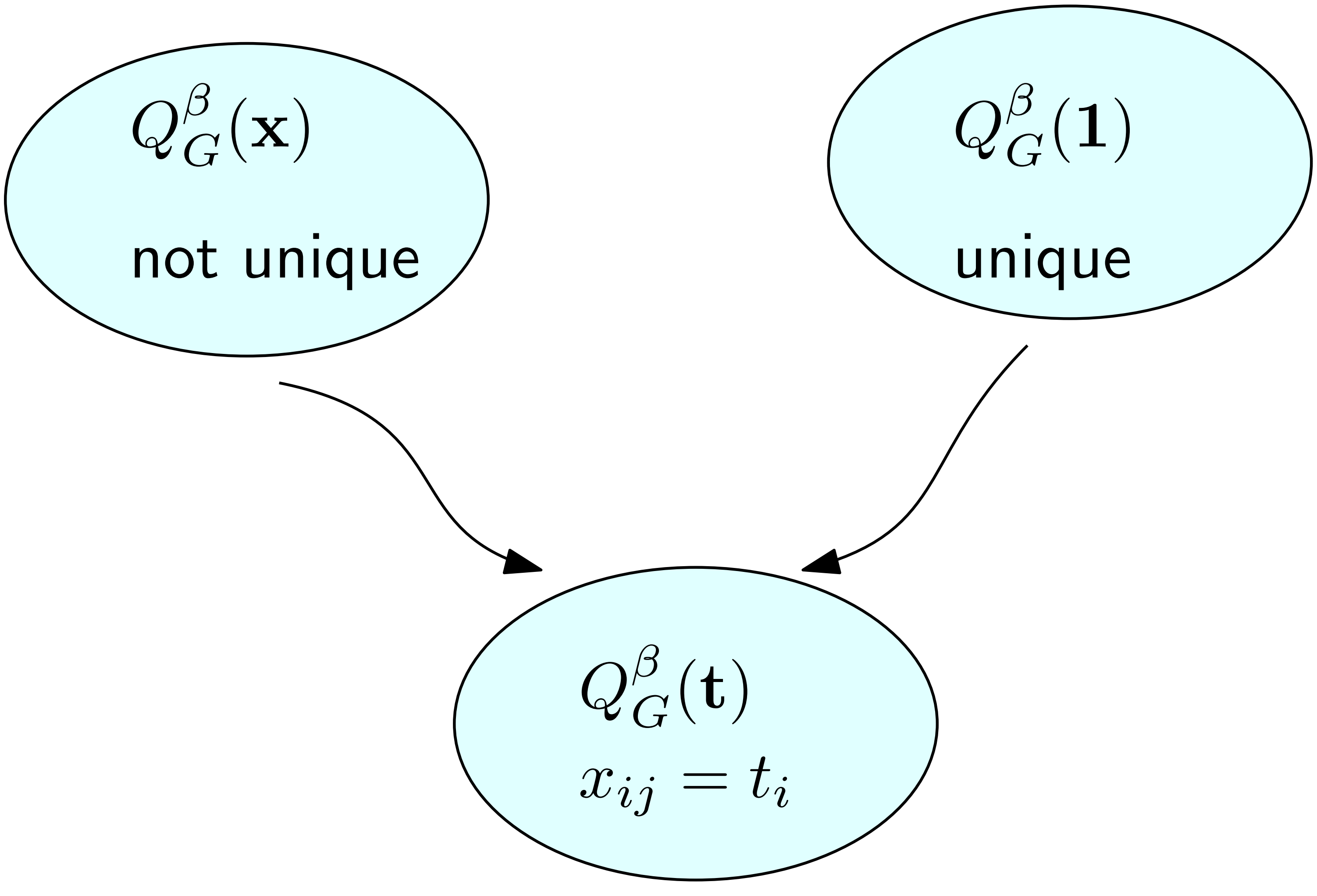
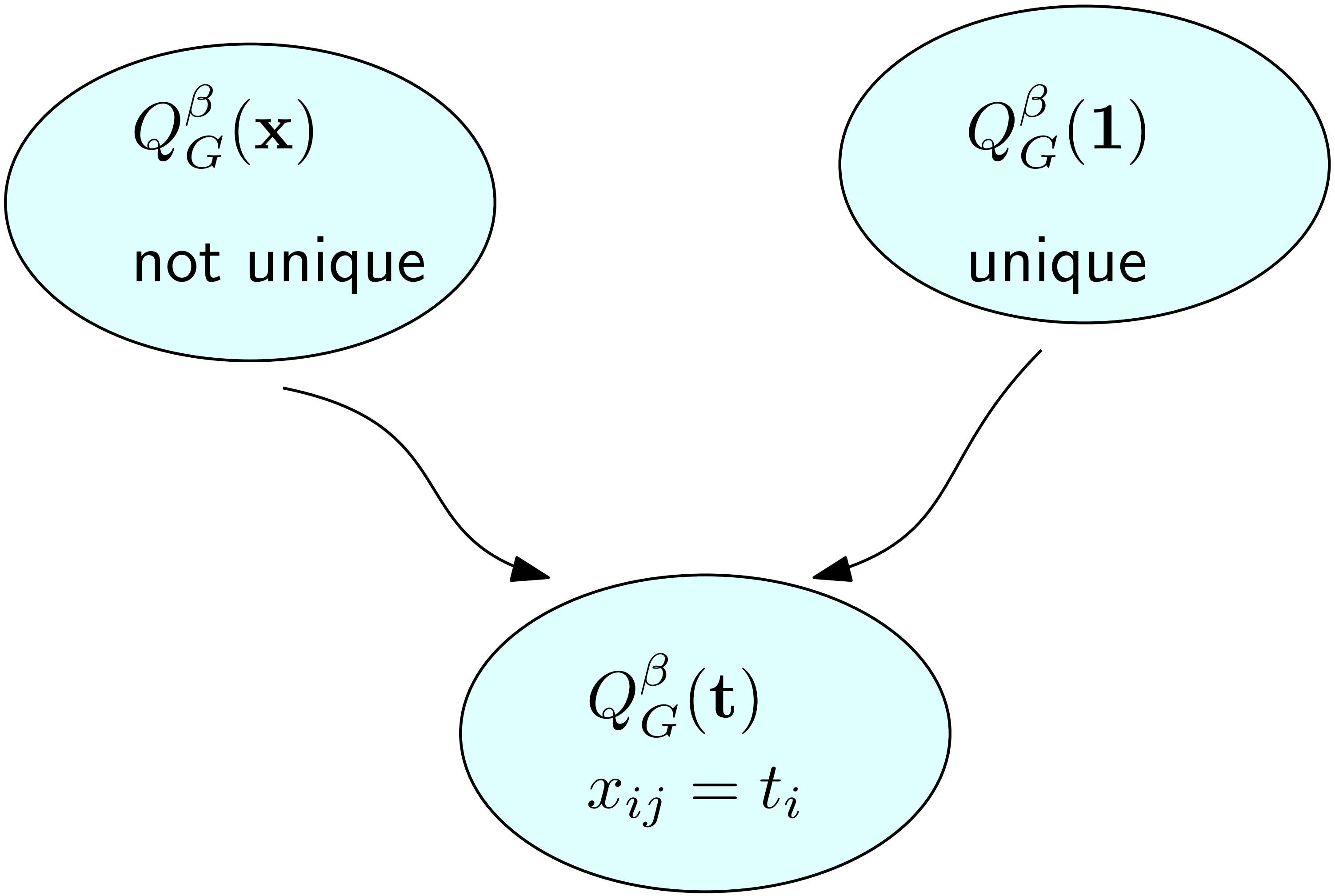
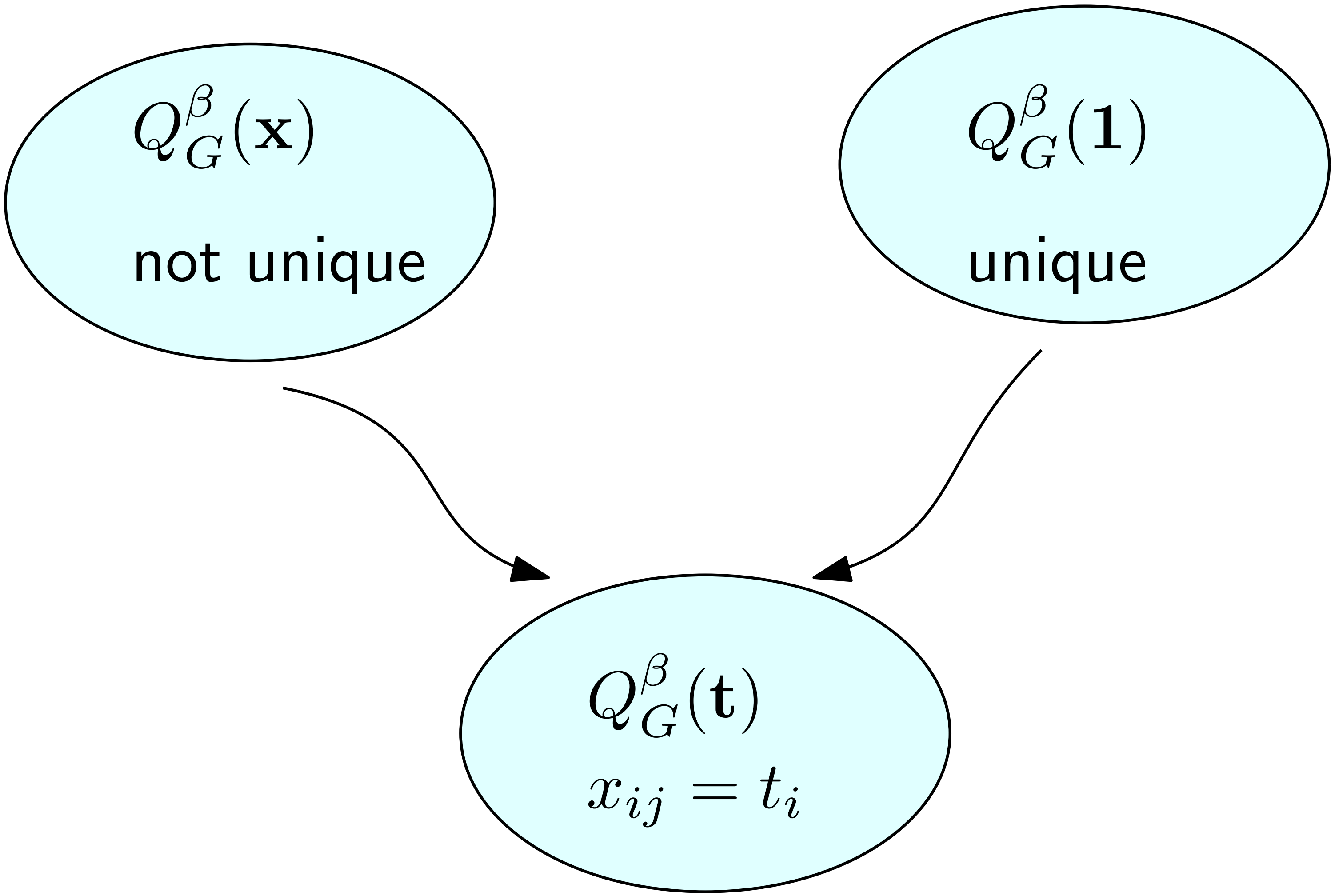


Uniqueness of reduced forms



Unique?

Uniqueness of reduced forms



Unique? Not unique?

Uniqueness of reduced forms

Conjecture. (M, 2015)

$Q_G^\beta(\mathbf{t})$

$x_{ij} = t_i$

is unique.

Uniqueness of reduced forms

Conjecture. (M, 2015)

$$Q_G^{\beta}(\mathbf{t}) \qquad x_{ij} = t_i$$

is independent of the order of reductions performed.

Uniqueness of reduced forms

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Theorem. (Grinberg, 2017)

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Uniqueness of reduced forms

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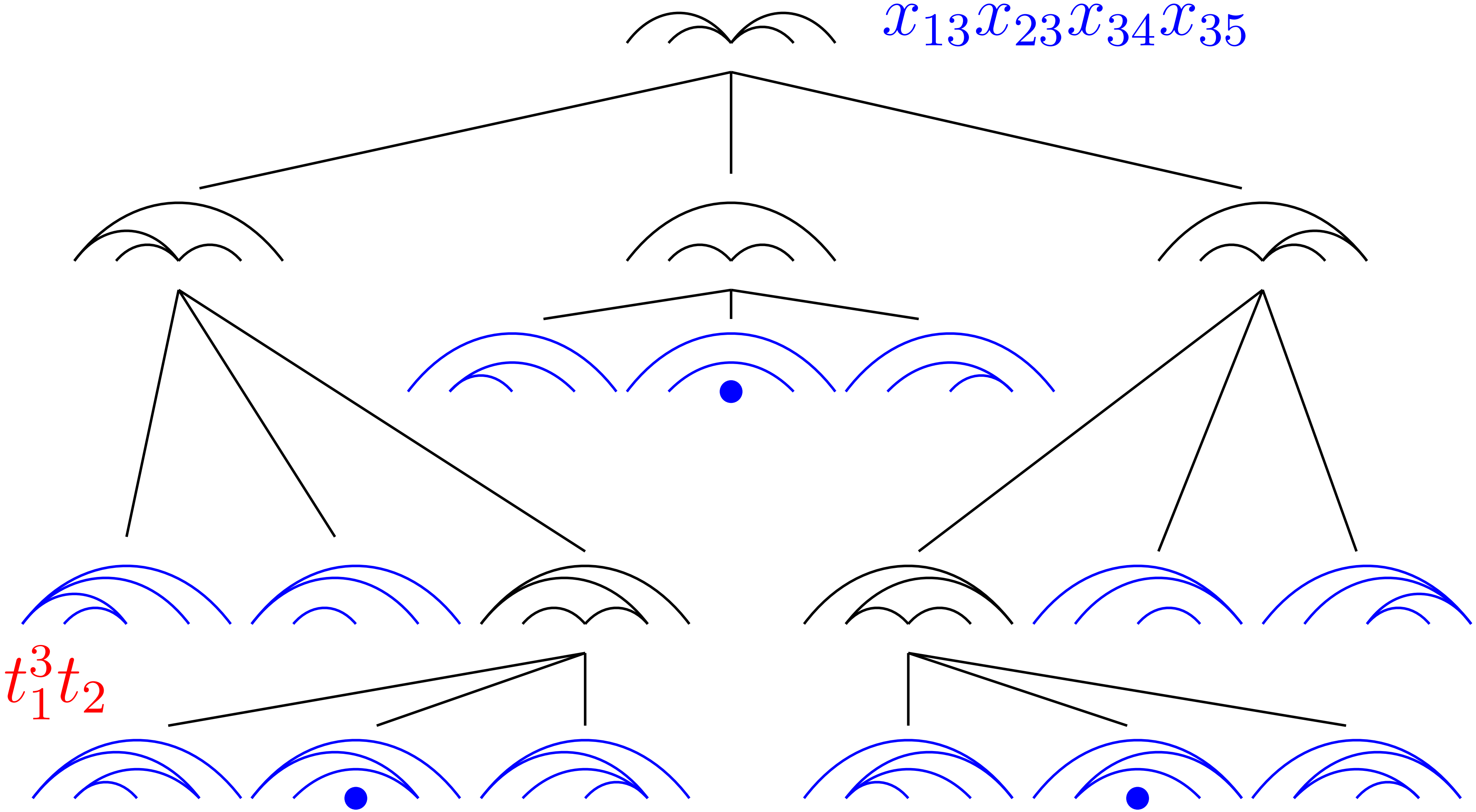
Theorem. (M, St. Dizier, 2017)

A combinatorial description of

$Q_G^\beta(\mathbf{t})$

$x_{ij} = t_i$

Reduced form $Q_G^{\beta}(\mathbf{t})$



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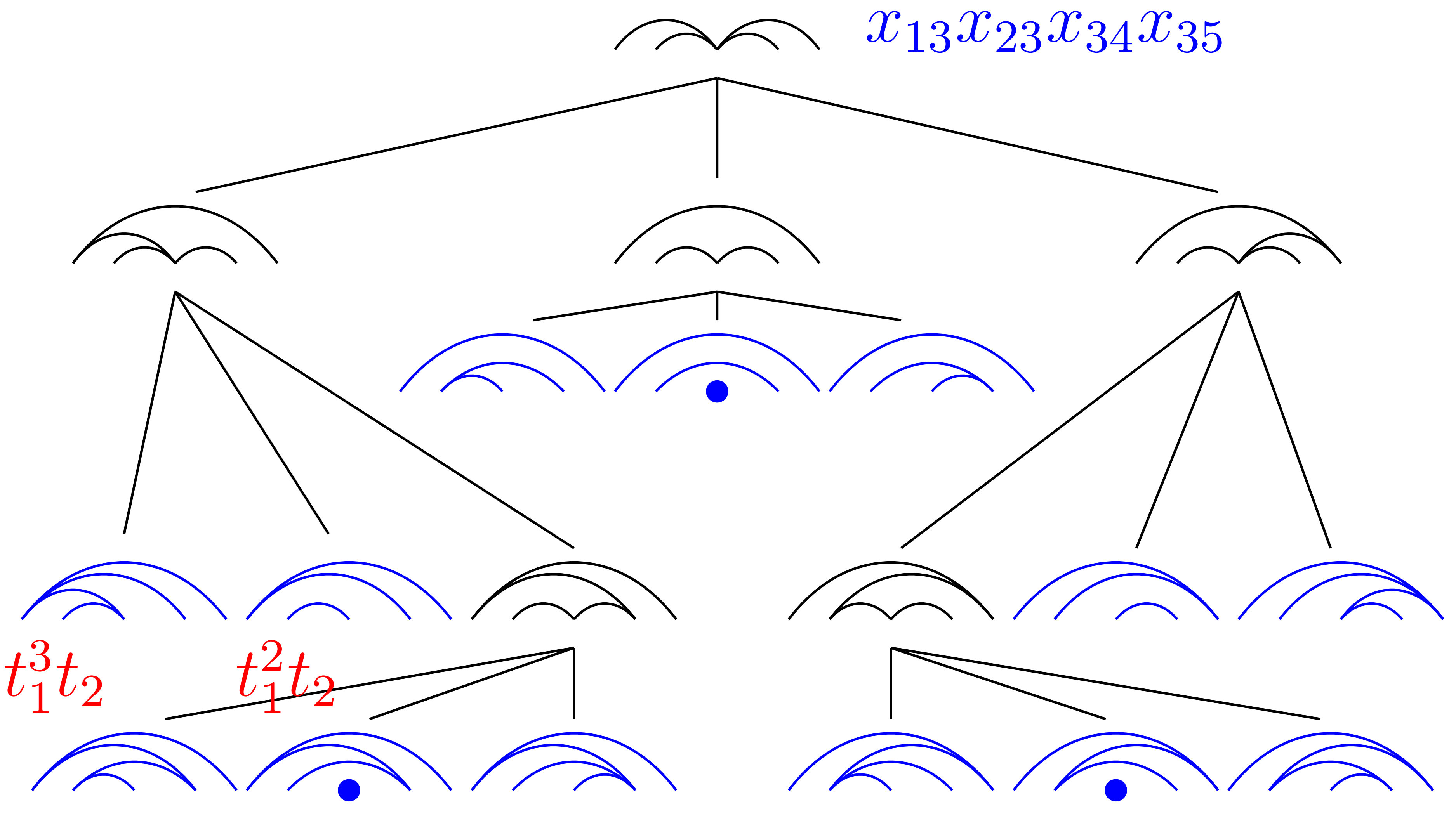

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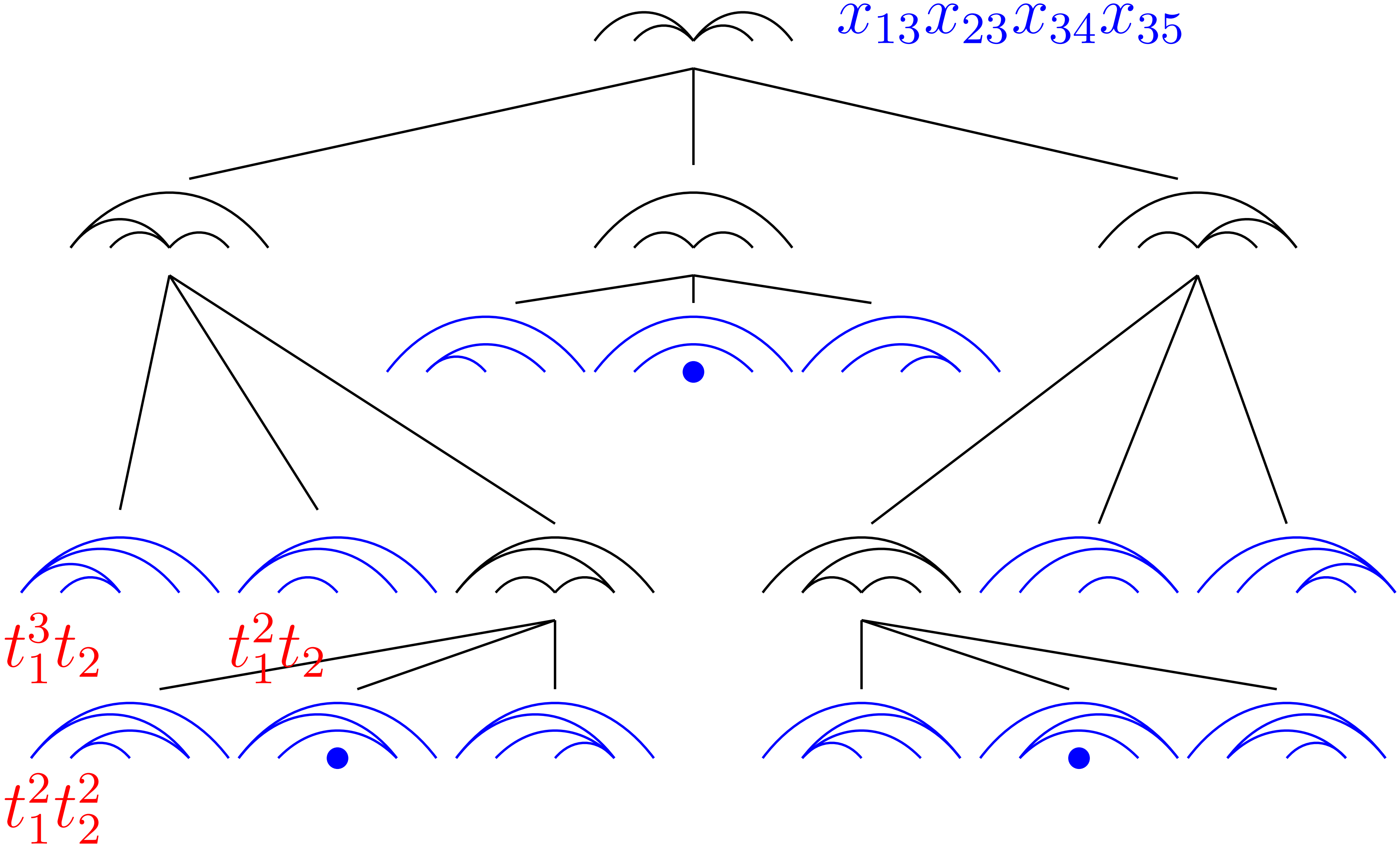
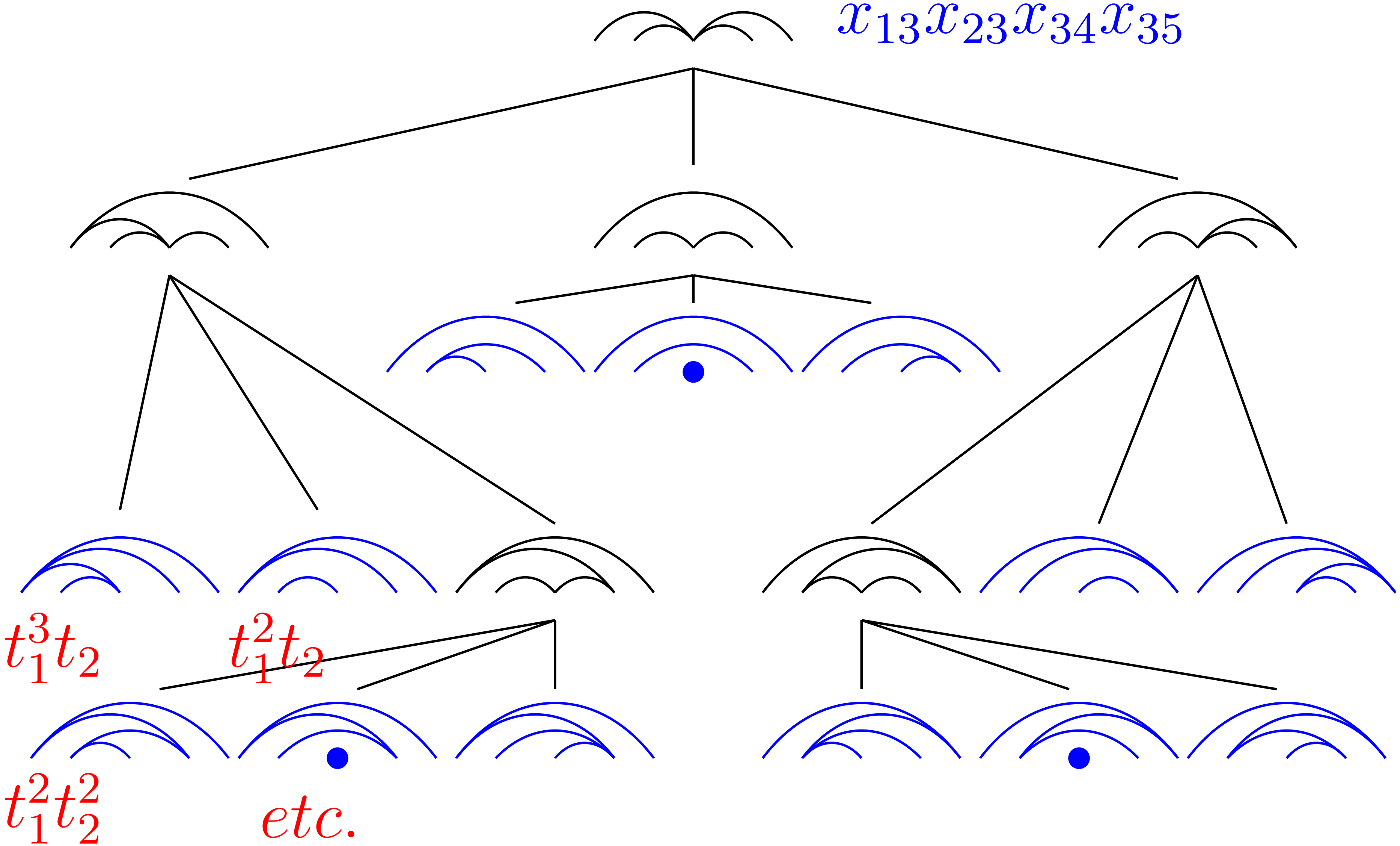
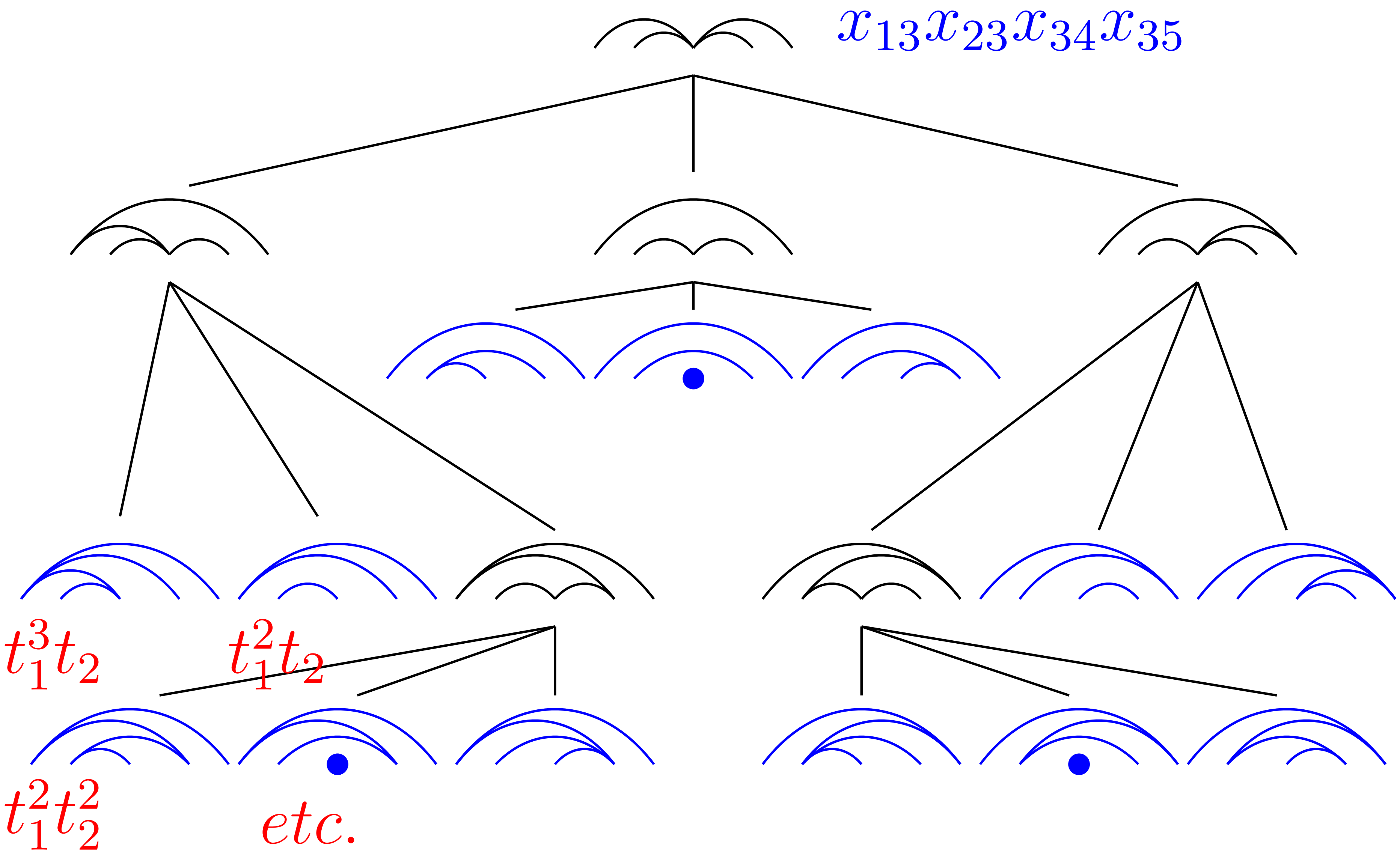


Reduced form $Q_G^{\beta}(\mathbf{t})$



Reduced form $Q^{\beta}_{G}(\mathbf{t})$



We are encoding the right degrees (RD) of the leaves of the reduction tree.

Grothendieck polynomials are t-reduced forms

Theorem (Escobar, M., 2015)

Given $\pi = 1\pi'$, π' dominant, we have that

$$Q^{\beta}_{T(\pi)}(\mathbf{t}) = \left(\prod_{i=1}^{n-1} t_i^{g_i}\right)\mathfrak{G}^{\beta}_{\pi^{-1}}(t_1^{-1}, \ldots, t_{n-1}^{-1}). \quad \star$$

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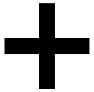
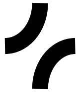
Theorem. (M, St. Dizier, 2017)

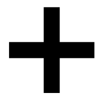
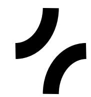
For $\beta = -1$ the polynomial $Q^{\beta}_{G}(\mathbf{t})$ is a weighted integer point enumerator of the Newton polytope of $Q^{\beta}_{G}(\mathbf{t})$, with nonzero weights.

Moreover, the exponents of the homogeneous pieces of $Q^{\beta}_{G}(\mathbf{t})$ are integer points of generalized permutahedra.

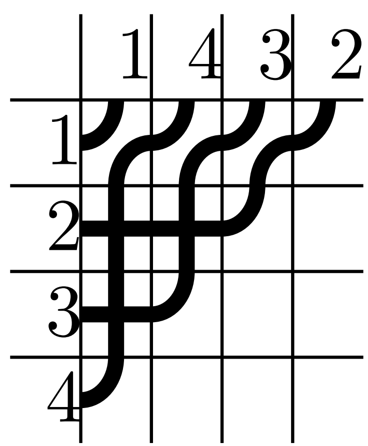
Towards Grothendiecks

A pipe dream for $\pi \in S_n$ is a tiling of an $n \times n$ matrix with two tiles crosses  and elbows  such that

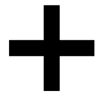
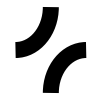
Towards Grothendiecks

A pipe dream for $\pi \in S_n$ is a tiling of an $n \times n$ matrix with two tiles crosses  and elbows  such that

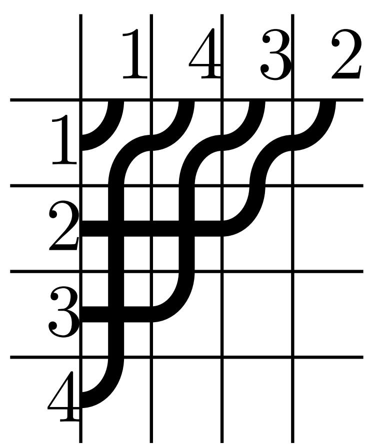
- all tiles in the weak south-east triangle are elbows, and
  (they are not drawn on the figure!)



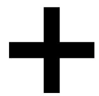
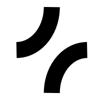
Towards Grothendiecks

A pipe dream for $\pi \in S_n$ is a tiling of an $n \times n$ matrix with two tiles crosses  and elbows  such that

- all tiles in the weak south-east triangle are elbows, and
  (they are not drawn on the figure!)
- if we write $1, 2, \ldots, n$ on the left and follow the strands (ignoring second crossings among the same strands) they come out on the top and read π.

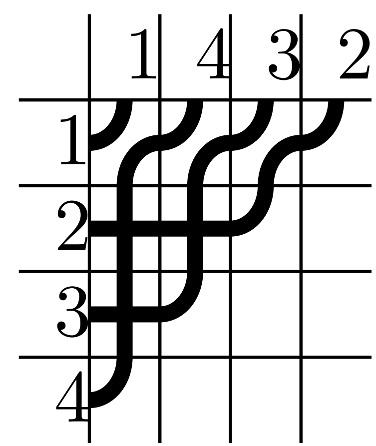


Towards Grothendiecks

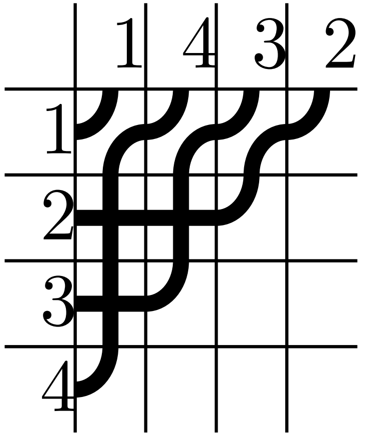
A pipe dream for $\pi \in S_n$ is a tiling of an $n \times n$ matrix with two tiles crosses  and elbows  such that

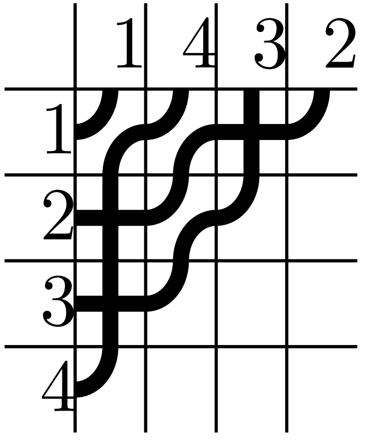
- all tiles in the weak south-east triangle are elbows, and (they are not drawn on the figure!)
- if we write $1, 2, \ldots, n$ on the left and follow the strands (ignoring second crossings among the same strands) they come out on the top and read π.

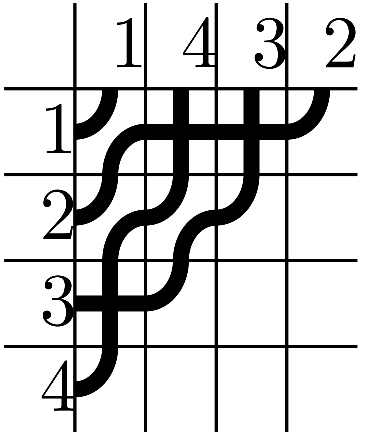
A pipe dream is reduced if no two strands cross twice.

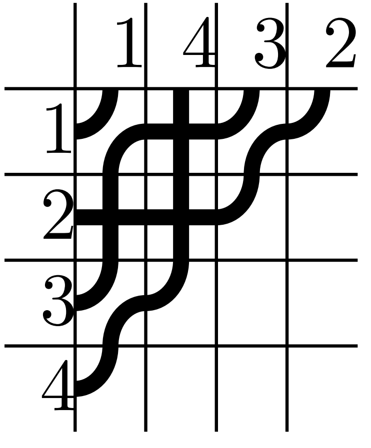


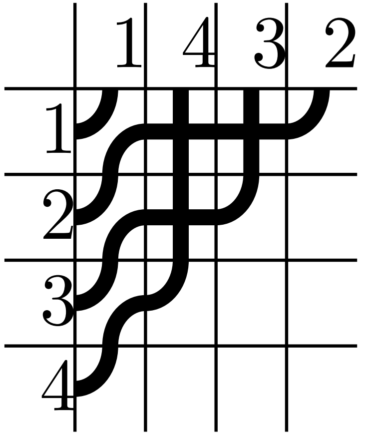
Reduced pipe dreams



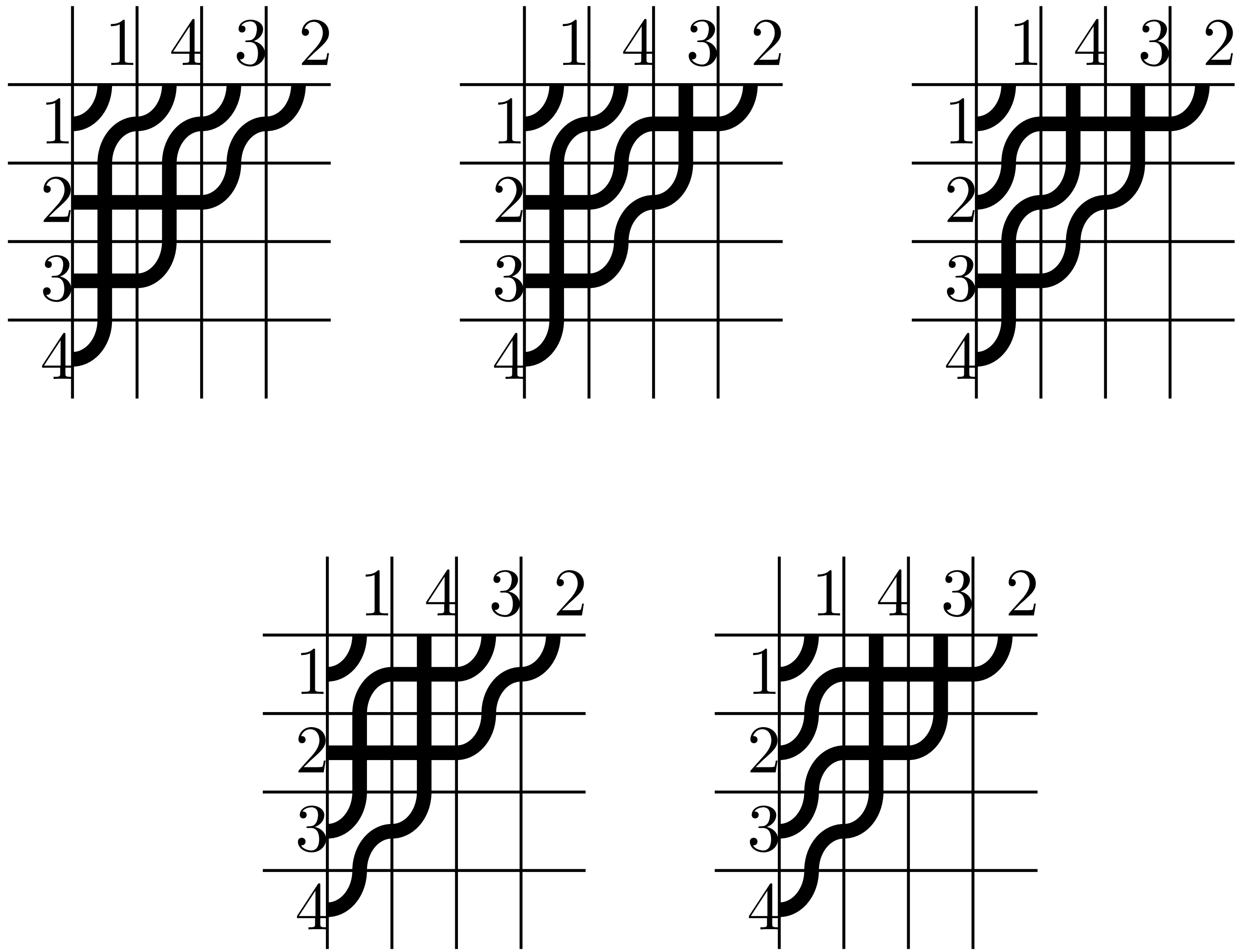








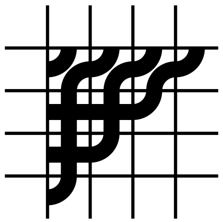
Reduced pipe dreams



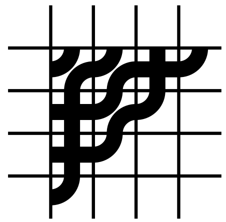
Bergeron-Billey: ladder and chute moves connect these!

Pipe dreams of 1432

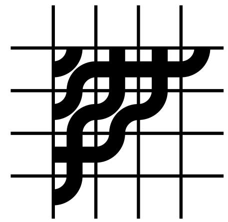
Reduced pipe dreams of 1432 (with 3 crosses)



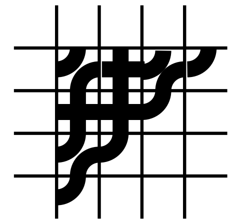
$x_2^2 x_3$



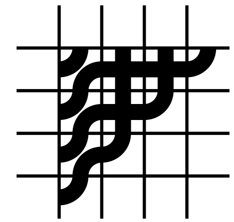
$x_1 x_2 x_3$



$x_1^2 x_3$



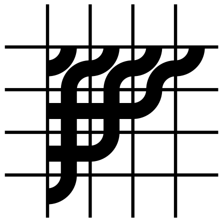
$x_1 x_2^2$



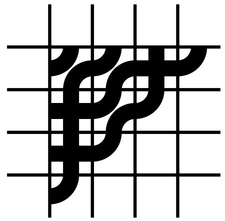
$x_1^2 x_2$

Pipe dreams of 1432

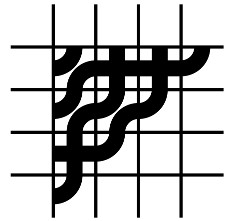
Reduced pipe dreams of 1432 (with 3 crosses)



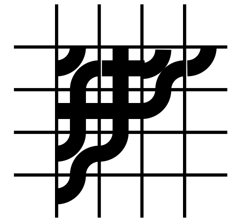
$x_2^2 x_3$



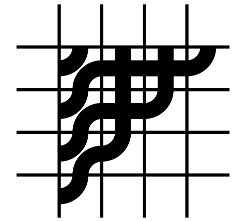
$x_1 x_2 x_3$



$x_1^2 x_3$

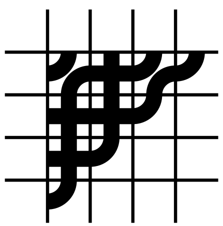


$x_1 x_2^2$

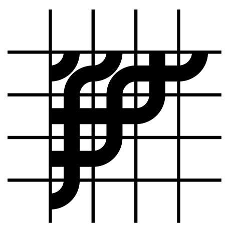


$x_1^2 x_2$

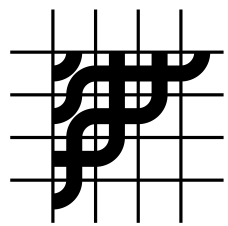
Nonreduced pipe dreams of 1432 with 4 crosses



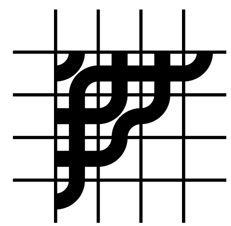
$x_1 x_2^2 x_3$



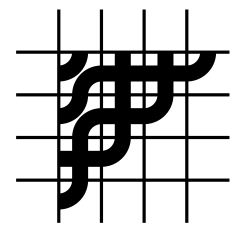
$x_1 x_2^2 x_3$



$x_1^2 x_2 x_3$



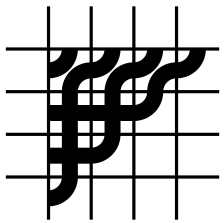
$x_1^2 x_2 x_3$



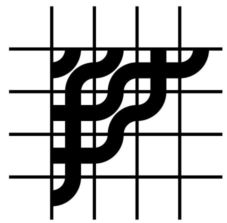
$x_1^2 x_2 x_3$

Pipe dreams of 1432

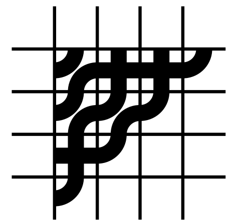
Reduced pipe dreams of 1432 (with 3 crosses)



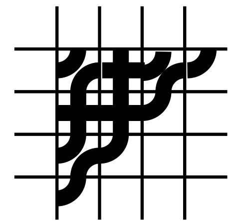
$x_2^2x_3$



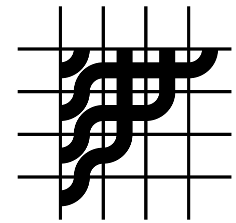
$x_1x_2x_3$



$x_1^2x_3$

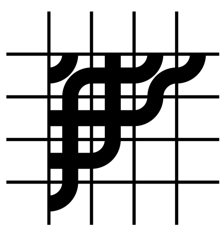


$x_1x_2^2$

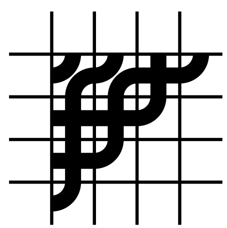


$x_1^2x_2$

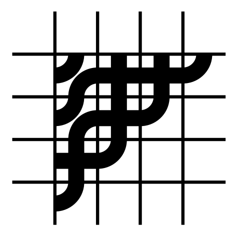
Nonreduced pipe dreams of 1432 with 4 crosses



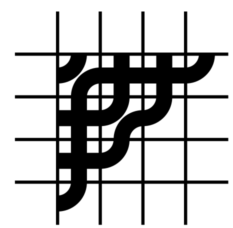
$x_1x_2^2x_3$



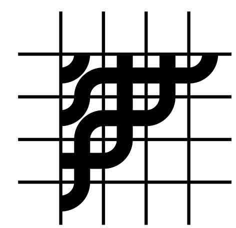
$x_1x_2^2x_3$



$x_1^2x_2x_3$

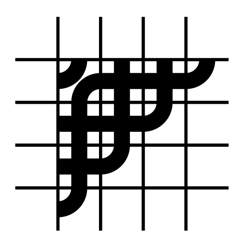


$x_1^2x_2x_3$



$x_1^2x_2x_3$

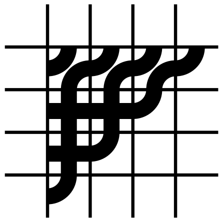
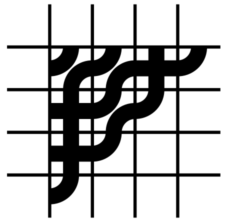
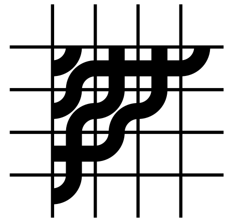
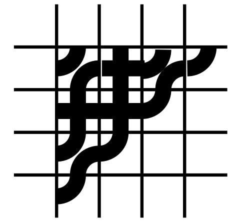
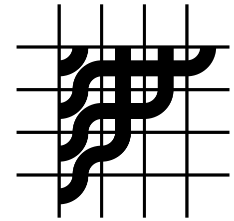
Nonreduced pipe dreams of 1432 with 5 crosses



$x_1^2x_2^2x_3$

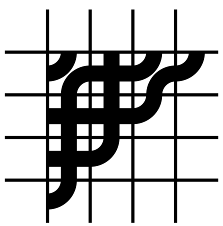
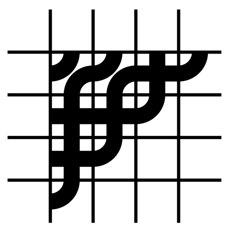
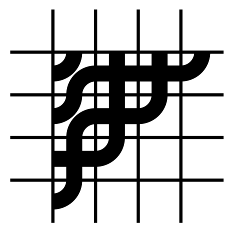
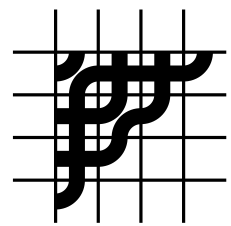
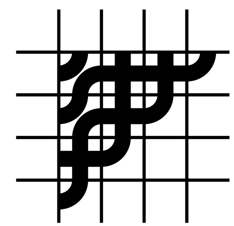
Grothendieck polynomial of 1432

Reduced pipe dreams of 1432 (with 3 crosses)

$\mathfrak{G}_w(\mathbf{x}) = \quad x_2^2x_3 \quad x_1x_2x_3 \quad x_1^2x_3 \quad x_1x_2^2 \quad x_1^2x_2$

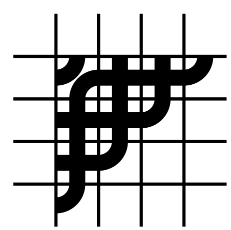
Nonreduced pipe dreams of 1432 with 4 crosses

$x_1x_2^2x_3 \quad x_1x_2^2x_3 \quad x_1^2x_2x_3 \quad x_1^2x_2x_3 \quad x_1^2x_2x_3$

Nonreduced pipe dreams of

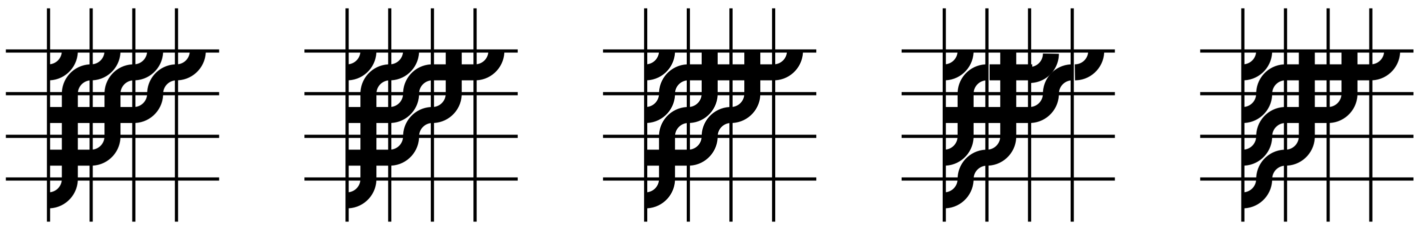
1432 with 5 crosses



$x_1^2x_2^2x_3$

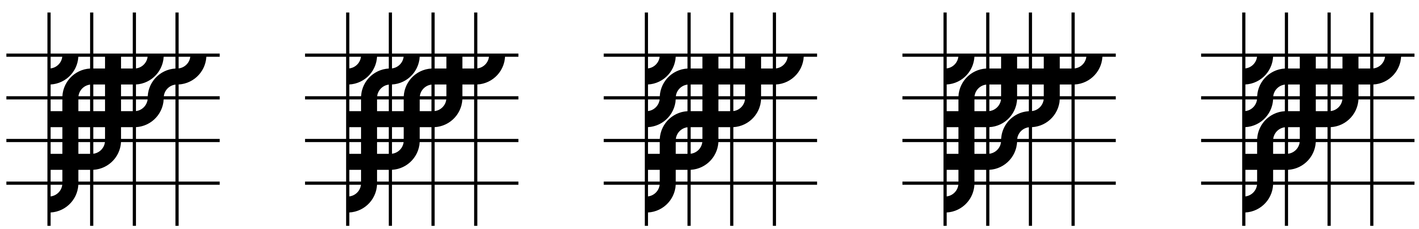
Grothendieck polynomial of 1432

Reduced pipe dreams of 1432 (with 3 crosses)



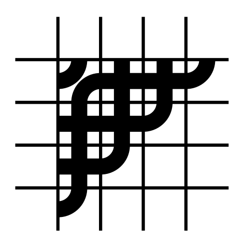
$$\mathfrak{G}_w(\mathbf{x}) = x_2^2x_3 + \quad x_1x_2x_3 + \quad x_1^2x_3 + \quad x_1x_2^2 + \quad x_1^2x_2$$

Nonreduced pipe dreams of 1432 with 4 crosses



$x_1x_2^2x_3$ $\quad x_1x_2^2x_3$ $\quad x_1^2x_2x_3$ $\quad x_1^2x_2x_3$ $\quad x_1^2x_2x_3$

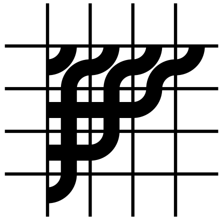
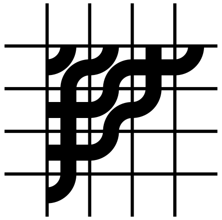
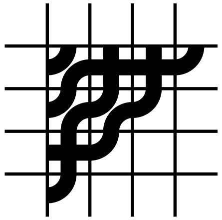
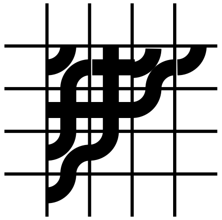
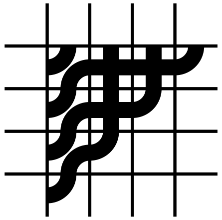
Nonreduced pipe dreams of 1432 with 5 crosses



$x_1^2x_2^2x_3$

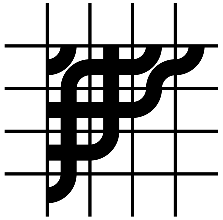
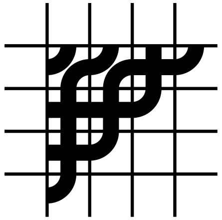
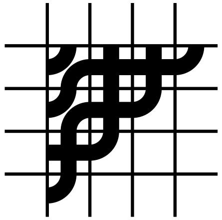
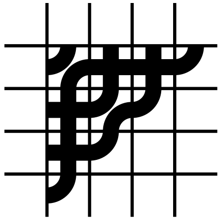
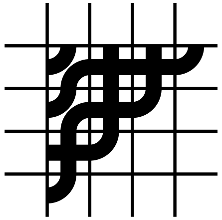
Grothendieck polynomial of 1432

Reduced pipe dreams of 1432 (with 3 crosses)

$$\mathfrak{G}_w(\mathbf{x}) = x_2^2 x_3 + \quad x_1 x_2 x_3 + \quad x_1^2 x_3 + \quad x_1 x_2^2 + \quad x_1^2 x_2$$

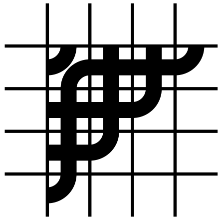
Nonreduced pipe dreams of 1432 with 4 crosses

$$+(-1)\Big(x_1 x_2^2 x_3 + \quad x_1 x_2^2 x_3 + \quad x_1^2 x_2 x_3 + \quad x_1^2 x_2 x_3 + \quad x_1^2 x_2 x_3 \Big)$$

Nonreduced pipe dreams of

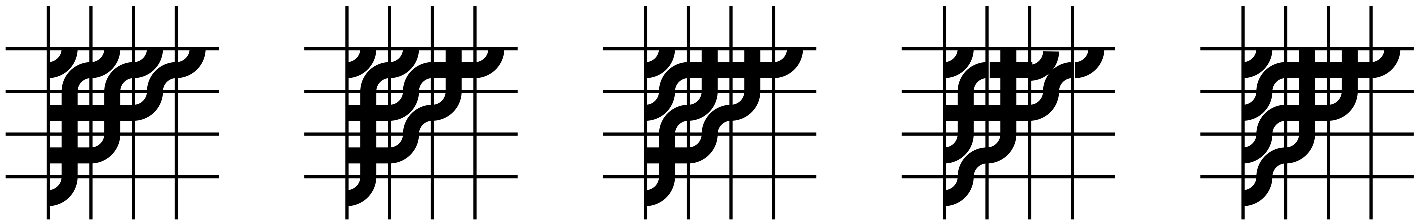
1432 with 5 crosses



$x_1^2 x_2^2 x_3$

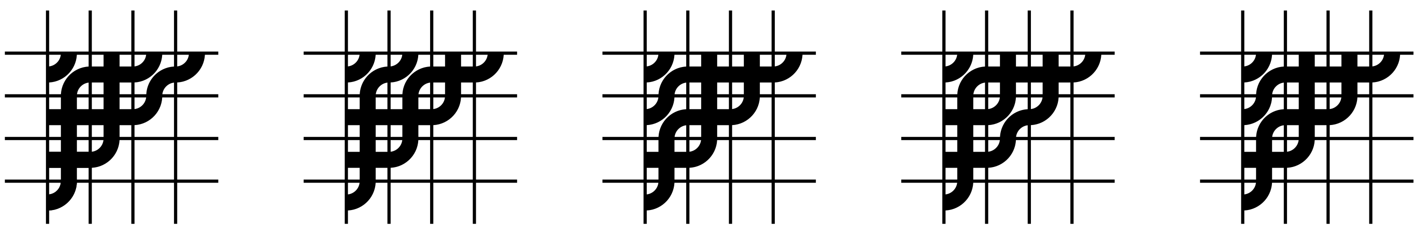
Grothendieck polynomial of 1432

Reduced pipe dreams of 1432 (with 3 crosses)



$$\mathfrak{G}_w(\mathbf{x}) = x_2^2x_3 + \quad x_1x_2x_3 + \quad x_1^2x_3 + \quad x_1x_2^2 + \quad x_1^2x_2$$

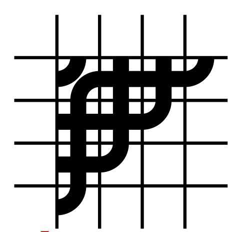
Nonreduced pipe dreams of 1432 with 4 crosses



$$+(-1)\Big(x_1x_2^2x_3 + \quad x_1x_2^2x_3 + \quad x_1^2x_2x_3 + \quad x_1^2x_2x_3 + \quad x_1^2x_2x_3\Big)$$

Nonreduced pipe dreams of

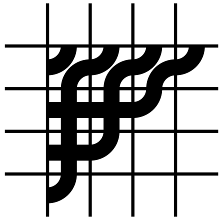
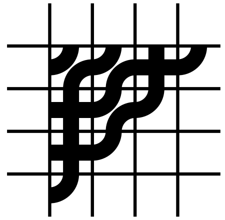
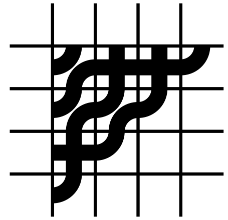
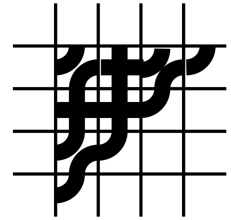
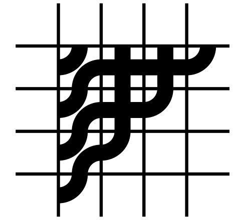
1432 with 5 crosses



$$+(-1)^2\Big(x_1^2x_2^2x_3\Big)$$

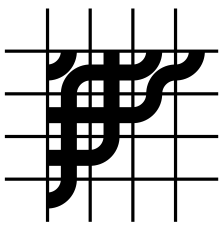
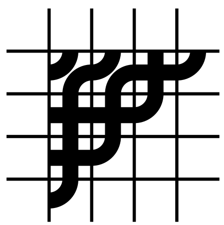
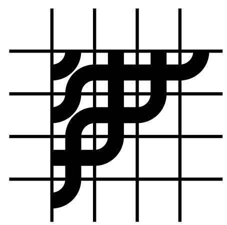
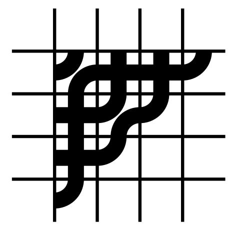
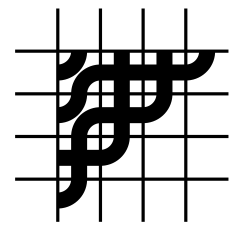
β-Grothendieck polynomial of 1432

Reduced pipe dreams of 1432 (with 3 crosses)

$\mathfrak{G}_w^{\beta}(\mathbf{x}) =$ $x_2^2x_3$ $x_1x_2x_3$ $x_1^2x_3$ $x_1x_2^2$ $x_1^2x_2$

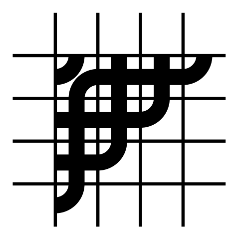
Nonreduced pipe dreams of 1432 with 4 crosses

$x_1x_2^2x_3$ $x_1x_2^2x_3$ $x_1^2x_2x_3$ $x_1^2x_2x_3$ $x_1^2x_2x_3$

Nonreduced pipe dreams of

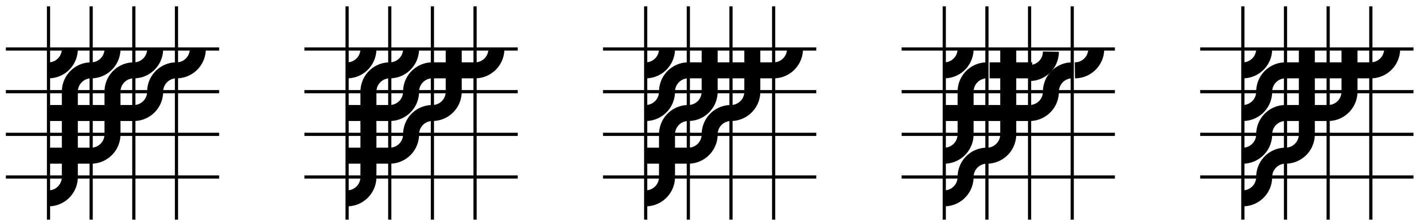
1432 with 5 crosses



$x_1^2x_2^2x_3$

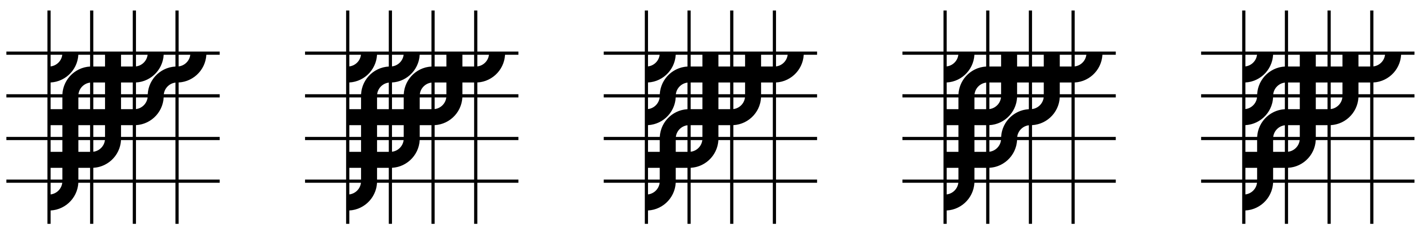
β-Grothendieck polynomial of 1432

Reduced pipe dreams of 1432 (with 3 crosses)



$$\mathfrak{G}_w^{\beta}(\mathbf{x}) = x_2^2x_3 + \quad x_1x_2x_3 + \quad x_1^2x_3 + \quad x_1x_2^2 + \quad x_1^2x_2$$

Nonreduced pipe dreams of 1432 with 4 crosses



$x_1x_2^2x_3$ $\quad x_1x_2^2x_3$ $\quad x_1^2x_2x_3$ $\quad x_1^2x_2x_3$ $\quad x_1^2x_2x_3$

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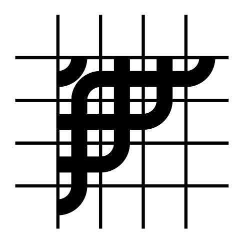

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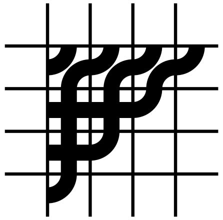
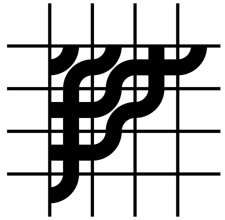
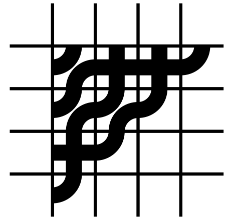
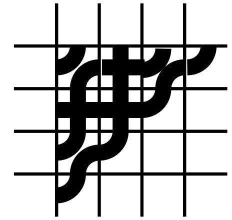
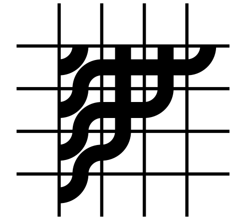
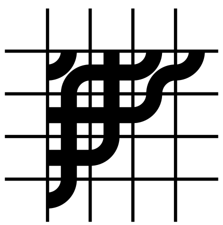
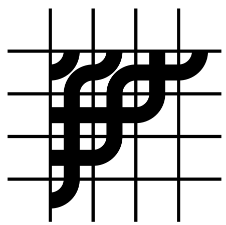
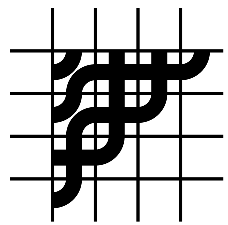
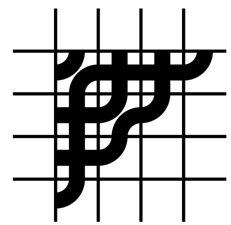
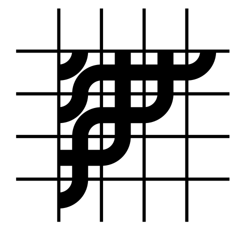
    

$$\mathfrak{G}_w^{\beta}(\mathbf{x}) = x_2^2x_3 + \quad x_1x_2x_3 + \quad x_1^2x_3 + \quad x_1x_2^2 + \quad x_1^2x_2$$

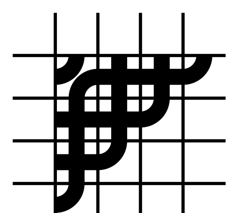
Nonreduced pipe dreams of 1432 with 4 crosses

$$+\beta\Big(x_1x_2^2x_3 + \quad x_1x_2^2x_3 + \quad x_1^2x_2x_3 + \quad x_1^2x_2x_3 + \quad x_1^2x_2x_3 \Big)$$

Nonreduced pipe dreams of

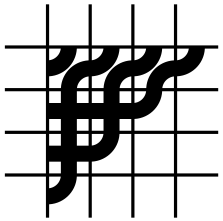
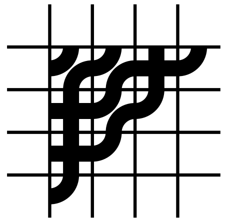
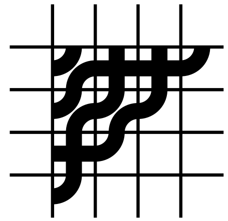
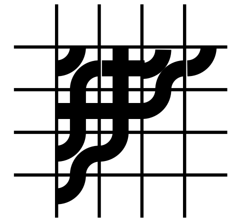
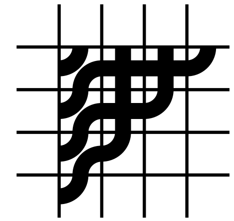
1432 with 5 crosses



$x_1^2x_2^2x_3$

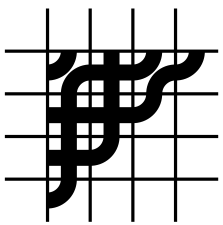
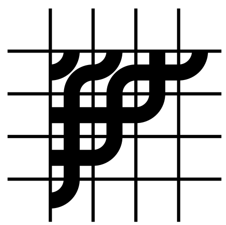
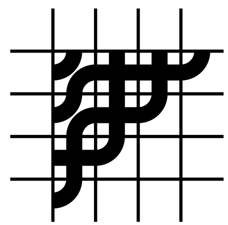
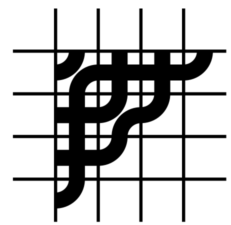
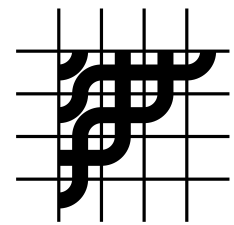
β-Grothendieck polynomial of 1432

Reduced pipe dreams of 1432 (with 3 crosses)

$$\mathfrak{G}_w^{\beta}(\mathbf{x}) = x_2^2x_3 + \quad x_1x_2x_3 + \quad x_1^2x_3 + \quad x_1x_2^2 + \quad x_1^2x_2$$

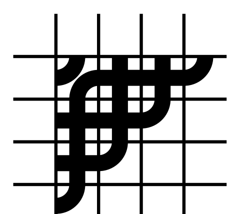
Nonreduced pipe dreams of 1432 with 4 crosses

$$+\beta\Big(x_1x_2^2x_3 + \quad x_1x_2^2x_3 + \quad x_1^2x_2x_3 + \quad x_1^2x_2x_3 + \quad x_1^2x_2x_3 \Big)$$

Nonreduced pipe dreams of

1432 with 5 crosses



$$+\beta^2\Big(x_1^2x_2^2x_3 \Big)$$

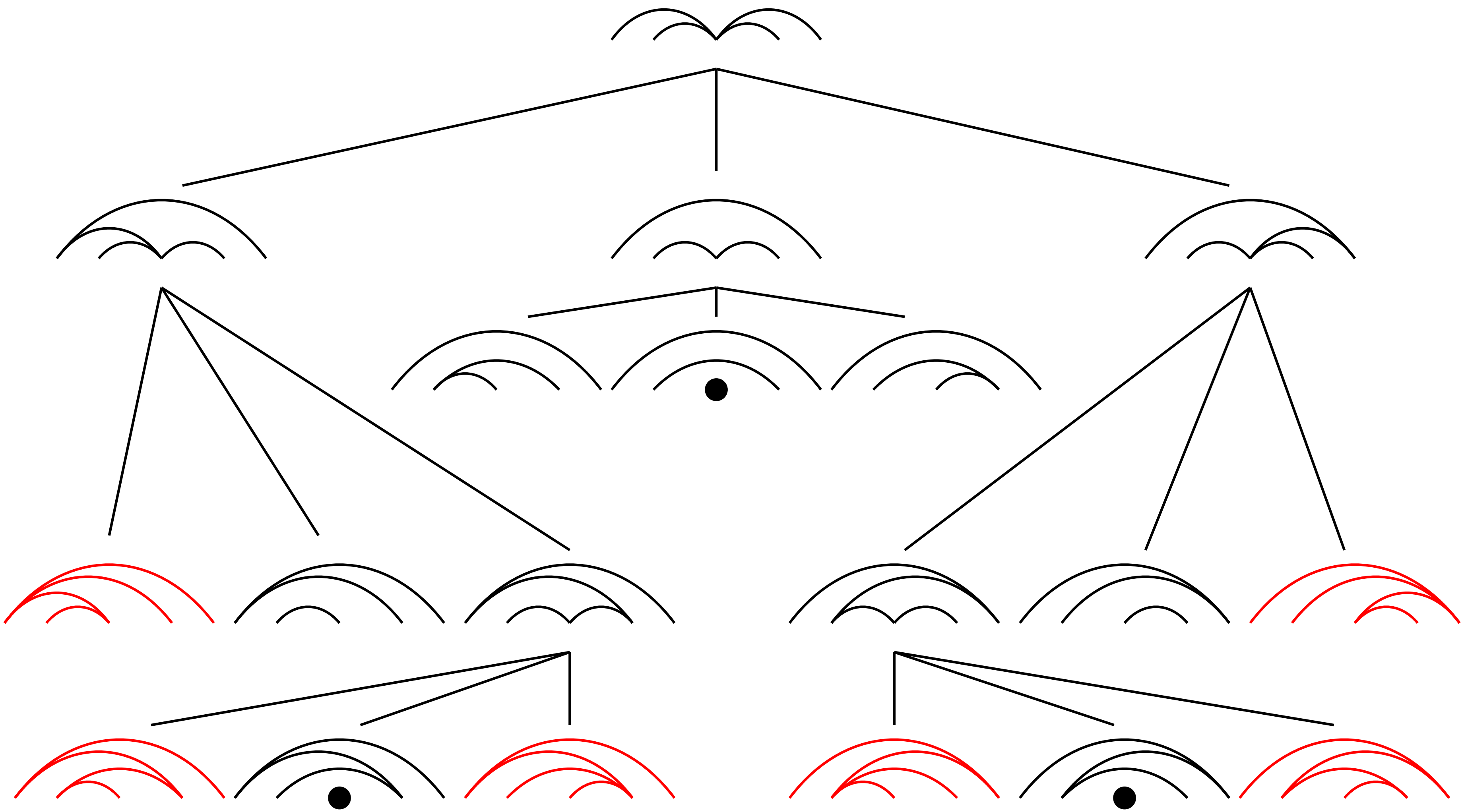
Grothendieck polynomials are t-reduced forms

Theorem (Escobar, M., 2015)

Given $\pi = 1\pi'$, π' dominant, we have that

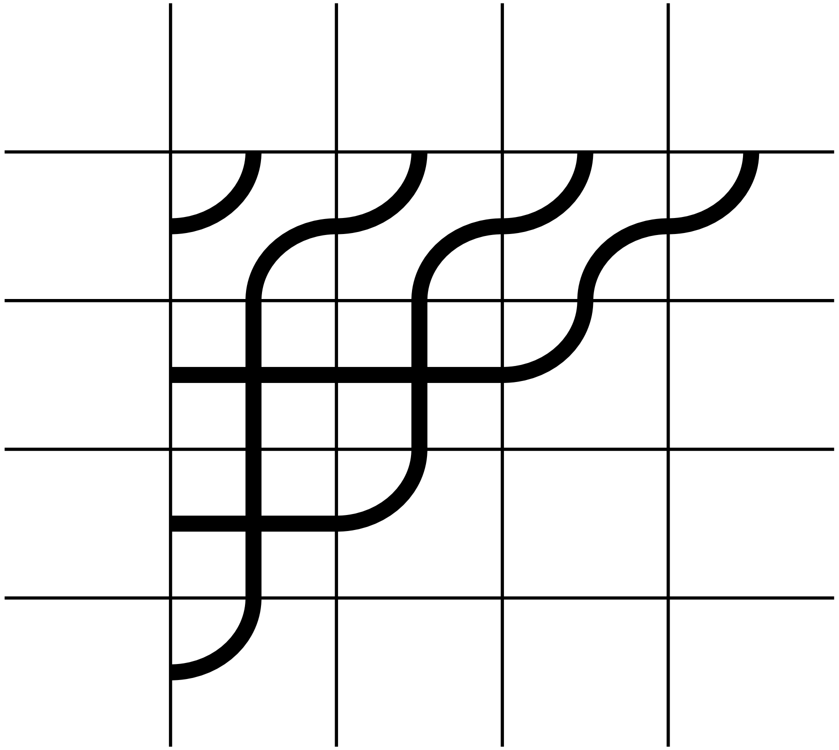
$$Q^{\beta}_{T(\pi)}(\mathbf{t}) = \left(\prod_{i=1}^{n-1} t_i^{g_i}\right)\mathfrak{G}^{\beta}_{\pi^{-1}}(t_1^{-1}, \ldots, t_{n-1}^{-1}). \quad \star$$

Canonical reduction tree

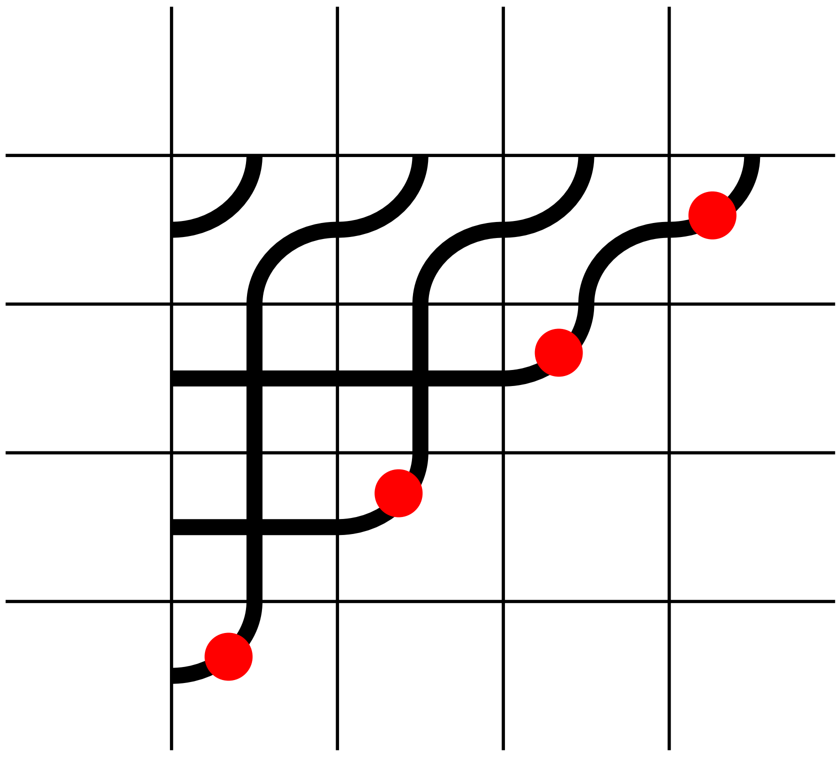


Theorem (M, 2009) The full dimensional leaves of the canonical reduction tree are the noncrossing alternating spanning trees of the directed transitive closure of the noncrossing tree at the root.

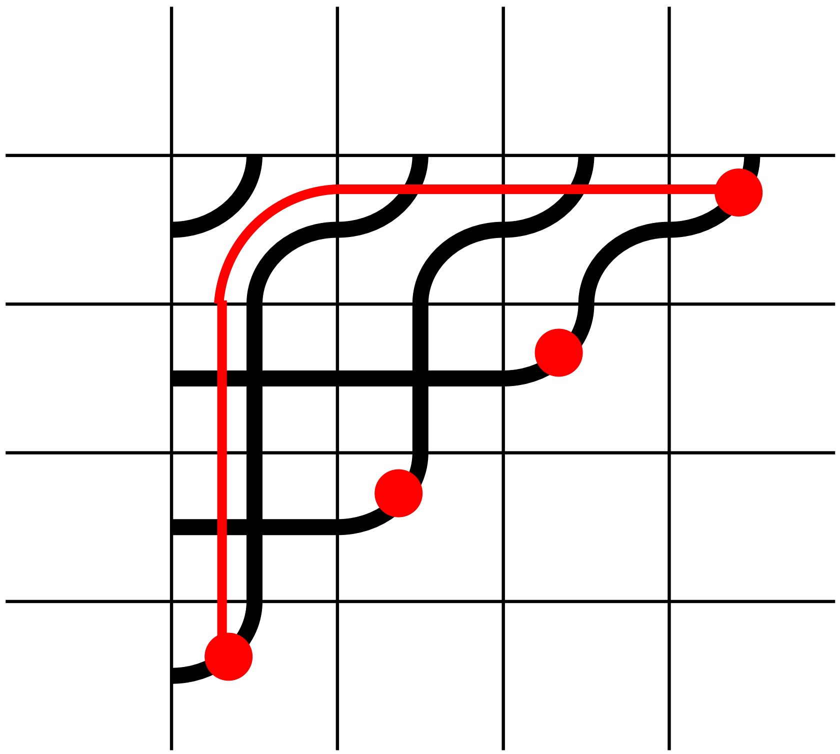
Pipe dream to alternating tree



Pipe dream to alternating tree



Pipe dream to alternating tree



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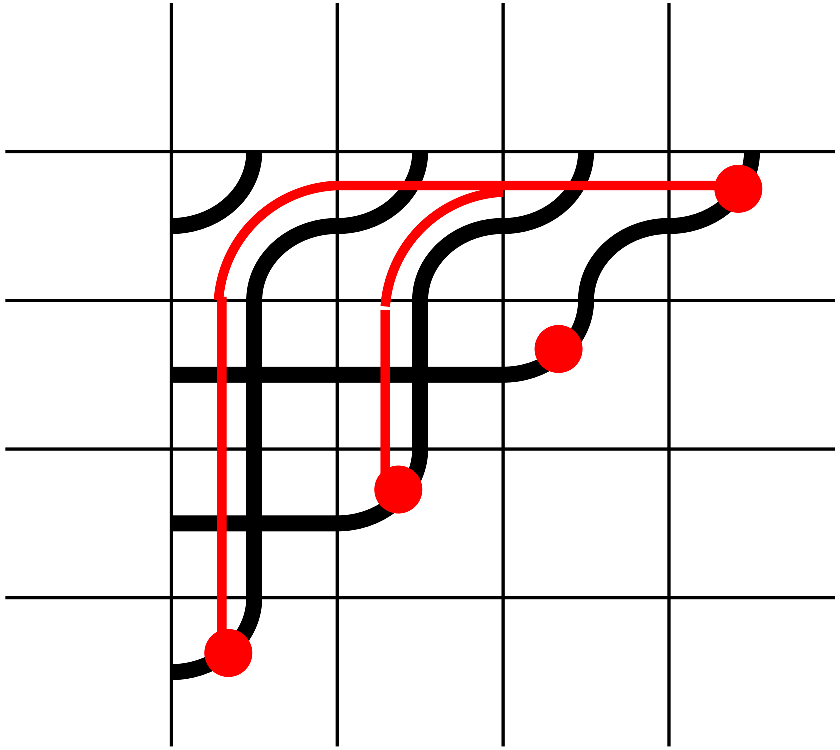
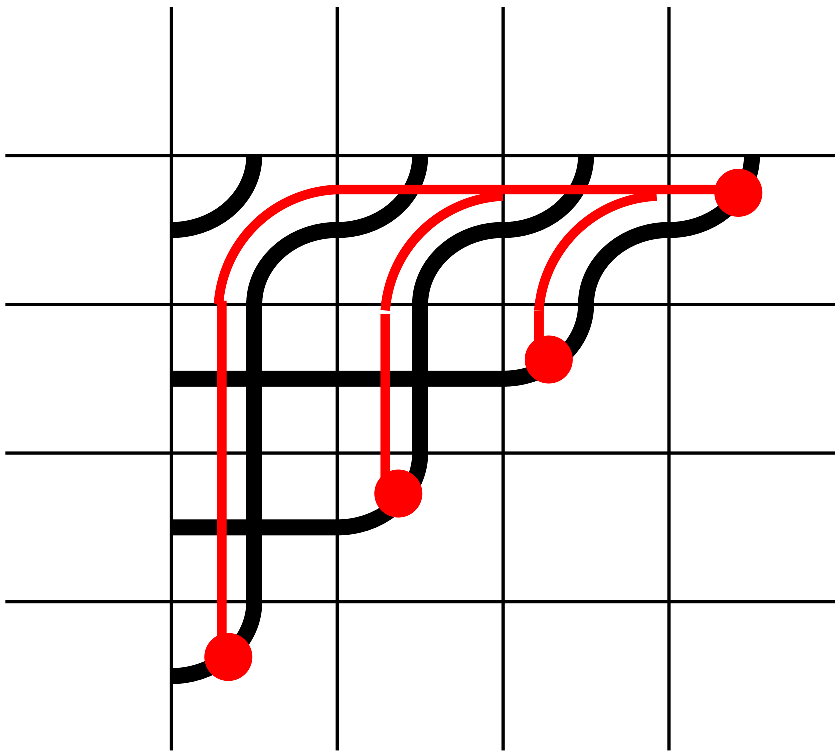
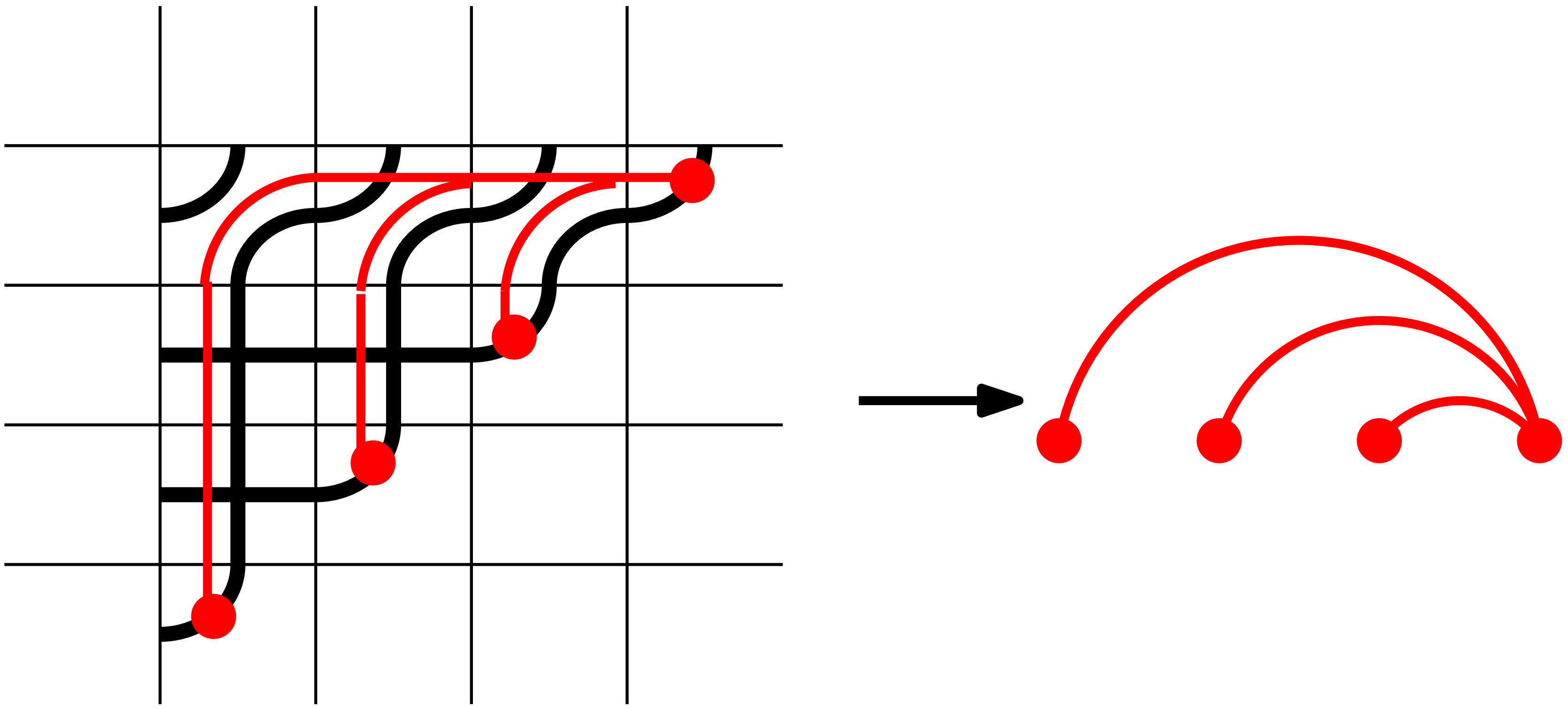


Pipe dream to alternating tree



Pipe dream to alternating tree



Flow polytopes and...

- Kostant partition functions ✓

- Grothendieck polynomials ✓

Flow polytopes and...

- Kostant partition functions ✓
- Grothendieck polynomials ✓
- space of diagonal harmonics

Diagonal harmonics and Tesler matrices

Haglund-Loehr 2005, Carlsson-Mellit 2015

$$\mathrm{Hilb}(\mathrm{DH_n}; \mathrm{q}, \mathrm{t}) = \sum_{\pi \text{ parking function of } n} \mathrm{q}^{\mathrm{dinv}(\pi)} \mathrm{t}^{\mathrm{area}(\pi)}$$

Diagonal harmonics and Tesler matrices

Haglund-Loehr 2005, Carlsson-Mellit 2015

$$\mathrm{Hilb}(\mathrm{DH_n}; \mathrm{q}, \mathrm{t}) = \sum_{\pi \text{ parking function of } n} \mathrm{q}^{\mathrm{dinv}(\pi)} \mathrm{t}^{\mathrm{area}(\pi)}$$

Theorem (Haglund 2011)

$$\mathrm{Hilb}(\mathrm{DH_n}; \mathrm{q}, \mathrm{t}) = \sum_{\text{Tesler matrices } \mathrm{A}} \mathrm{wt_{q,t}}(\mathrm{A})$$

Alternant and Tesler matrices

Theorem (Garsia-Haglund 2002)

$$C_n(q,t) := \mathrm{Hilb}(\mathrm{DH}_{\mathrm{n}}^{\epsilon};\mathrm{q},\mathrm{t}) = \sum_{\pi \text{ Dyck paths size } n} \mathrm{q}^{\mathrm{area}(\pi)}\mathrm{t}^{\mathrm{bounce}(\pi)}$$

Alternant and Tesler matrices

Theorem (Garsia-Haglund 2002)

$$C_n(q,t) := \mathrm{Hilb}(\mathrm{DH}_\mathrm{n}^{\epsilon}; \mathrm{q}, \mathrm{t}) = \sum_{\pi \text{ Dyck paths size } n} \mathrm{q}^{\mathrm{area}(\pi)} \mathrm{t}^{\mathrm{bounce}(\pi)}$$

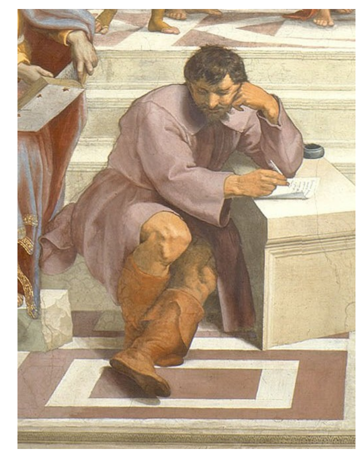
Theorem (Gorsky-Negut 2013)

$$C_n(q,t) = \sum_{\text{Tesler matrices A}} wt'_{q,t}(A)$$

Tesler matrices are the integer points of the Tesler polytope, which is a flow polytope of the complete graph (M-Morales-Rhoades 2014).

- (with A. H. Morales) Volumes and Ehrhart polynomials of flow polytopes, **arxiv:1710.00701**
- (with A. St. Dizier) From generalized permutahedra to Grothendieck polynomials via flow polytopes, **arxiv:1705.02418**
- (with L. Escobar), Subword complexes via triangulations of root polytopes. *Algebraic Combinatorics (2018)*
- (with A. H. Morales and B. Rhoades) The polytope of Tesler matrices. *Selecta Mathematica (2017)*

Panta Rhei = everything flows (Heraclitus)



Thank you!