Lovász Local Lemma – a new tool to asymptotic enumeration?

Linyuan Lincoln Lu
László Székely
University of South Carolina

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Overview

- LLL and its generalizations
- LLL – an instance of the Poisson paradigm
- New negative dependency graphs
- Applications:
  - Permutation enumeration
  - Latin rectangle enumeration
  - Regular graph enumeration
- A joke

When none of the events happen

- Assume that $A_1, A_2, \ldots, A_n$ are events in a probability space Ω. How can we infer $\bigcap_{i=1}^{n} \overline{A_i} \neq \varnothing$?
- If A_i's are mutually independent, $P(A_i)<1$, then $P\left(\bigcap_{i=1}^{n} \overline{A_i}\right) = \prod_{i=1}^{n} P\left(\overline{A_i}\right) = \prod_{i=1}^{n}\left(1 - P\left(A_i\right)\right) > 0$
- If $\sum_{i=1}^{n} P\left(A_i\right) < 1$, then

$$P\left(\bigcap_{i=1}^{n} \overline{A_i}\right) = P\left(\overline{\bigcup_{i=1}^{n} A_i}\right) = 1 - P\left(\bigcup_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} P\left(A_i\right) > 0$$

A way to combine arguments:

- Assume that $A_1, A_2, \ldots, A_n$ are events in a probability space Ω.
- Graph G is a dependency graph of the events $A_1, A_2, \ldots, A_n$, if $V(G)=\{1,2,\ldots,n\}$ and each A_i is independent of the elements of the event algebra generated by

$$\left\{A_j : ij \notin E(G)\right\}$$

Lovász Local Lemma (Erdős-Lovász 1975)

- Assume G is a dependency graph for $A_1, A_2, \ldots, A_n$, and d=max degree in G
- If for i=1,2,…,n, $P(A_i) < p$, and $e(d+1)p < 1$, then

$$P\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$$

Lovász Local Lemma (Spencer)

- Assume G is a dependency graph for $A_1, A_2, \ldots, A_n$
- If there exist $x_1, x_2, \ldots, x_n$ in $[0,1)$ such that

$$P(A_i) \le x_i \prod_{ij \in E(G)} (1 - x_j)$$

then

$$P\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \ge \prod_{i=1}^{n} (1 - x_i) > 0$$

Negative dependency graphs

- Assume that $A_1, A_2, \ldots, A_n$ are events in a probability space Ω.
- Graph G with $V(G)=\{1,2,\ldots,n\}$ is a negative dependency graph for events $A_1, A_2, \ldots, A_n$, if $\forall i \forall S \subseteq \{j : ij \notin E(G)\}$ $P\left(\bigcap_{j\in S} \overline{A_j}\right) > 0$ implies $P\left(A_i \middle| \bigcap_{j\in S} \overline{A_j}\right) \leq P(A_i)$

LLL: Erdős-Spencer 1991, Albert-Freeze-Reed 1995, Ku

- Assume G is a negative dependency graph for $A_1, A_2, \ldots, A_n$, exist $x_1, x_2, \ldots, x_n$ in $[0,1)$ such that, $P(A_i) \leq x_i \prod_{ij \in E(G)} (1 - x_j)$, then

$$P\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \geq \prod_{i=1}^{n} (1 - x_i) > 0$$

- Setting $x_i = 1/(d+1)$ implies the uniform version both for dependency and negative dependency

Needle in the haystack

- LLL has been in use for existence proofs to exhibit the existence of events of tiny probability. Is it good for other purposes?
- Where to find negative dependency graphs that are not dependency graphs?

Poisson paradigm

- Assume that $A_1, A_2, \ldots, A_n$ are events in a probability space Ω, $p(A_i)=p_i$. Let X denote the sum of indicator variables of the events. If dependencies are rare, X can be approximated with Poisson distribution of mean $\Sigma\, p_i$.
- X~Poisson means $P(X=k)=e^{-\mu}\mu^k/k!$ using $k=0$,

$$P\left(\bigcap_{i=1}^{n}\overline{A_i}\right)\approx e^{-\mu}=e^{-\sum_{i=1}^{n}p_i}$$

Models for the Poisson paradigm

- Chen-Stein method 1975-78
- Janson inequality 1990
- Brun's sieve
- Now: LLL. Assume G is negative dependency graph, $0<\varepsilon<0.14$.

$$\forall i : P(A_i) < \varepsilon; \sum_{ij\in E(G)} P(A_j) < \varepsilon \text{ imply } P\left(\bigcap_{j=1}^{n} \overline{A_j}\right) \geq e^{-(1+3\varepsilon)\sum_{j=1}^{n} P(A_j)}$$

Two negative dependency graphs

- H is a complete graph K_N or a complete bipartite graph $K_{N,L}$; Ω is the uniform probability space of maximal matchings in H. For a partial matching M, the canonical event $A_M = \{F \in \Omega \mid M \subseteq F\}$
- Canonical events A_M and A_{M^*} are in conflict: M and M^* have no common extension into maximal matching, i.e.

$$A_M \cap A_{M^*} = \varnothing$$

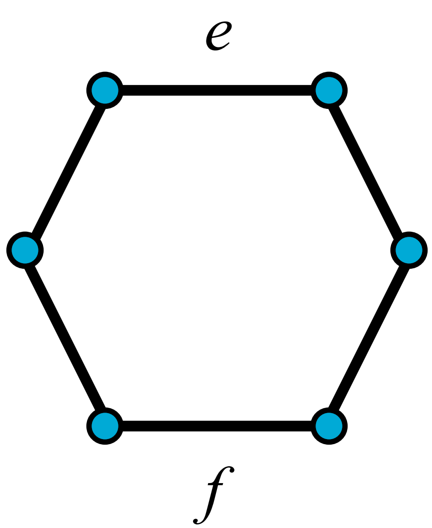
Main theorem

- For a graph $H=K_N$ or $K_{N,L}$, and a family of canonical events, if the edges of the graph G are defined by conflicts, then G is a negative dependency graph.

- This theorem fails to extend for the hexagon $H=C_6$

Hexagon example

- Two perfect matchings

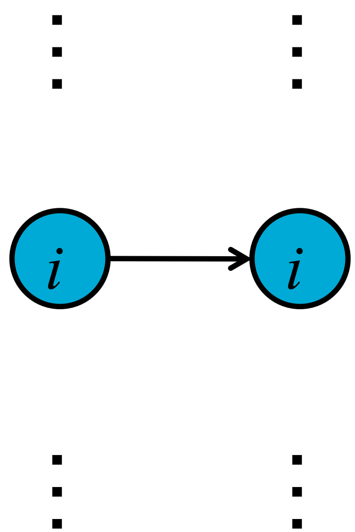


$$p\left(A_e\right) = p\left(A_f\right) = \tfrac{1}{2}$$

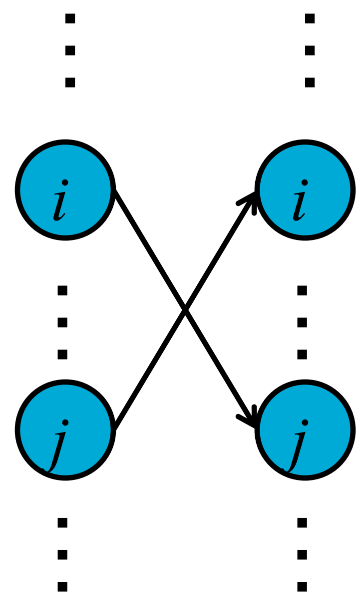
$$p\left(A_e \middle| \bar{A}_f\right) = \frac{p\left(A_e \cap \bar{A}_f\right)}{p\left(\bar{A}_f\right)} = \frac{\tfrac{1}{2}}{\tfrac{1}{2}} = 1 \not\leq p\left(A_e\right)$$

Relevance for permutation enumeration problems

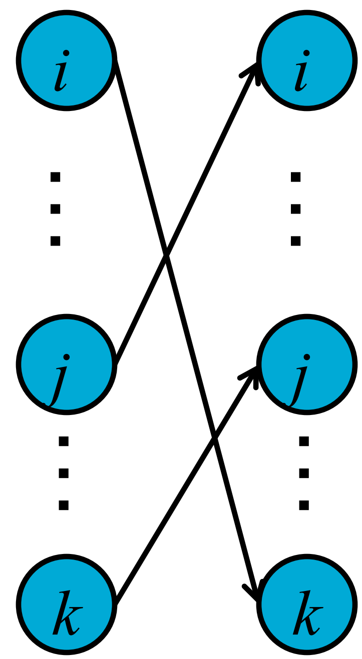
Derangements avoid:



2-cycle free avoids:



3-cycle free avoids:



ε-near-positive dependency graphs

- Assume that $A_1, A_2, \ldots, A_n$ are events in a probability space Ω.
- Graph G with $V(G)=\{1,2,\ldots,n\}$ is an ε– near-positive dependency graph of the events $A_1, A_2, \ldots, A_n$,
  - $ij \in E(G)$ implies $P\left(A_i \cap A_j\right) = 0$
  - $\forall i \forall S \subseteq \left\{j : ij \notin E(G)\right\}$

$$P\left(\bigcap_{j\in S} \overline{A_j}\right) > 0 \text{ implies } P\left(A_i \,\middle|\, \bigcap_{j\in S} \overline{A_j}\right) \geq (1-\varepsilon) P\left(A_i\right)$$

Quotient graphs

- Assume G is a negative dependency graph for $A_1, A_2, \ldots, A_n$. Assume further that $V(G)$ is partitioned into classes such that events in the same class are disjoint. For every partition class J, let $B_J = \bigcup_{j \in J} A_j$. The quotient graph of G is a negative dependency graph for the events B_J

Quotient graphs of ε-near-positive dependency graphs

- If the only edges of the quotient graph of an ε–near-positive dependency graph are loops, then the quotient graph is also an ε-near-positive dependency graph.

Asymptotic results

- A collection of matchings $\mathcal{M}$ is regular, if for every i, every vertex is covered d_i times by i-element matchings from $\mathcal{M}$
- A collection of matchings $\mathcal{M}$ is δ-sparse (details avoided!)
- Negative dependency graphs of δ-sparse collections of matchings are also ε-near-positive dependency graphs

Asymptotic results – a theorem

- A collection of matchings $\mathcal{M}$ in K_N or $K_{N,N}$ is regular, r is the largest matching size, $\mathcal{M}$ is δ-sparse. Set $\mu = \sum P(A_M)$ over $\mathcal{M}$. Suppose $\delta = o(\mu^{-1})$, μ is separated from 0, $\mu = o\left(\sqrt{N} r^{-3/2}\right)$ and $r = o\left(\sqrt{N}\right)$

Then $$P\left(\bigcap_M \overline{A_M}\right) = \left(1 + o(1)\right) e^{-\mu}$$

Consequences for permutation enumeration

- For k fixed, the proportion of k-cycle free permutations is $(1-o(1))e^{-1/k}$

- (Bender 70's) If $\max K$ grows slowly with n, the proportion of permutations free of cycles of length from set K is

$$(1-o(1))e^{-\sum_{k\in K} 1/k}$$

Latin rectangles

- Latin rectangle: k times n array filled with entries $1,2,\ldots,n$; putting a permutation into every row and not repeating an entry in any column ($k \leq n$)
- $L(k,n)$= number of k times n Latin rectangles

1	3	4	2	5
3	2	5	4	1
4	5	1	3	2

Enumeration of Latin rectangles

- $L(2,n) = n! \times$(# of derangements) $\approx (n!)^2 e^{-1}$
- Riordan 1944 $L(3,n) \approx (n!)^3 e^{-3}$
- Erdős-Kaplansky 1946

$$L(k,n) \sim (n!)^k e^{-\binom{k}{2}} \quad \text{for } k = o\left((\log n)^{3/2}\right)$$

- Yamamoto 1951 extended to $k = o\left(n^{1/3-\varepsilon}\right)$

Enumeration of Latin rectangles

- Stein 1978 (using Chen-Stein method)

$$L(k,n) \sim (n!)^k e^{-\binom{k}{2} - \frac{k^3}{6n}} \quad \text{for} \quad k = o\left(n^{1/2}\right)$$

- Godsil and McKay 1990 refined the asymptotics to make it work for

$$k = o\left(n^{6/7}\right)$$

Enumeration of Latin rectangles

- Skau 1990 (using van der Waerden's inequality for the permanent)

$$(n!)^k \prod_{r=1}^{k-1}\left(1-\frac{r}{n}\right)^n \le L(k,n)$$

and with this matched Stein's lower bound on a slightly smaller range:

$$L(k,n) \ge (1-o(1))(n!)^k e^{-\binom{k}{2}-\frac{k^3}{6n}}$$

for $k = o\left(n^{1/2}/\log n\right)$

Enumeration of Latin rectangles

- Quotient graph version of the negative dependency graph LLL yields Skau's lower bound:

$$(n!)^k \prod_{r=1}^{k-1} \left(1 - \frac{r}{n}\right)^n \le L(k,n)$$

matches the range of Stein's lower bound:

$$L(k,n) \ge (1 - o(1))(n!)^k e^{-\binom{k}{2} - \frac{k^3}{6n}}$$

for $k = o\left(n^{1/2} / \log n\right)$

Enumeration of Latin rectangles

- Quotient graph version of the near ε-positive dependency graph argument yields tight asymptotic upper bound in Yamamoto's range:

$$L(k,n) \le (1-o(1))(n!)^k e^{-\binom{k}{2}} \quad \text{for} \quad k = o\left(n^{\frac{1}{3}-\varepsilon}\right)$$

Relevance for Latin rectangle enumeration

1	3	4	2	5
3	2	5	4	1
4	5	1	3	2

- Try to fill in the fourth row with a permutation of [5].
- Complete bipartite graph: 1st class columns, 2nd class entries
- Canonical events defined by the edges 11, 13, 14; 23, 22, 25; 34, 35, 31; 42, 44, 43; 55, 51, 52

Enumeration of labeled regular graphs

- Bender-Canfield, independently Wormald 1978: d fix, nd even

$$\sqrt{2}e^{(1-d^2)/4}\left(\frac{d^d n^d}{e^d (d!)^2}\right)^{n/2}$$

Configuration model (Bollobás 1980)

- Put nd (nd even) vertices into n equal clusters
- Pick a random matching of K_{nd}
- Contract every cluster into a single vertex getting a multigraph or a simple graph
- Observe that all simple graphs are equiprobable

Enumeration of labeled regular graphs

- Bollobás 1980: nd even, $d < \sqrt{2 \log n}$

$$\left(1+o(1)\right)e^{\left(1-d^2\right)/4}\left(\frac{(dn-1)!!}{(d!)^n}\right)$$

- McKay 1985: for $d = o\left(n^{1/3}\right)$

Enumeration of labeled regular graphs

- McKay and Wormald: nd even, $d = o\left(n^{1/2}\right)$

$$\left(1+o(1)\right)e^{\left(1-d^2\right)/4-d^3/(12n)+O\left(d^2/n\right)}\left(\frac{(dn-1)!!}{(d!)^n}\right)$$

- Wormald 1981: fix $d \geq 3$, $g \geq 3$ girth

$$\left(1+o(1)\right)e^{-\sum_{i=1}^{g-1}\frac{(d-1)^i}{2i}}\frac{(dn-1)!!}{(d!)^n}$$

Theorem

- In the configuration model, if $d \geq 3$ and $g^3 d^{2g-3} = o(n)$, then the probability that the resulting random d-regular multigraph after the contraction has girth at least g, is

$$\left(1+o(1)\right)e^{-\sum_{i=1}^{g-1}\frac{(d-1)^i}{2i}}$$

hence the number of d-regular graphs with girth at least g is

$$\left(1+o(1)\right)e^{-\sum_{i=1}^{g-1}\frac{(d-1)^i}{2i}}\frac{(dn-1)!!}{(d!)^n}$$

Spencer's joke

- Spencer's joke in "Ten Lectures on the Probabilistic Method": uniformly selected random function from $[a]$ to $[b]$ is injection whp, if $b>>a^2$, but:
- LLL can show the existence of an injection when $2ea < b$.

Our joke

- Our joke: using negative dependency graph LLL, a uniformly selected random function $[a]$ to $[a]$ is injection with probability at least

$$\frac{a!}{a^a} > \left(\frac{1}{e} - o(1)\right)^a$$

- Combinatorial proof to the slightly weakened Stirling formula

Symmetric events

- Events $A_1, A_2, \ldots, A_n$ are symmetric, if the probability of their Boolean expressions do not change, when $A_{\pi(i)}$ is substituted for A_i simultaneously for any permutation π.

A Lemma for symmetric events

- If events $A_1, A_2, \ldots, A_n$ are symmetric

$$p_i = P\left(\bigcap_{j=1}^{i} \overline{A_j}\right) \quad \text{and} \quad p_0 = 1,$$

$$\forall i = 1,2,\ldots,n-1 \qquad p_i^2 \le p_{i-1} p_{i+1}$$

then LLL applies with empty negative dependency graph, $x_i = p_1$

Proof to our joke

- Consider a uniform random function f from $[a]$ to $[b]$ with $a \le b$. Let A_u denote the event that $f(u)$ occurs with multiplicity 2 or higher.

$$P(A_u) = 1 - b\frac{(b-1)^{a-1}}{b^a} \text{ and } p_i = i!\binom{b}{i}\frac{(b-i)^{a-i}}{b^a}$$

$$P(inj) \ge \left(1 - P(A_1)\right)^a = \left(1 - \frac{1}{a}\right)^{a(a-1)} = \left(\frac{1}{e} - o(1)\right)^a$$