Harmonic functions and the chromatic polynomial

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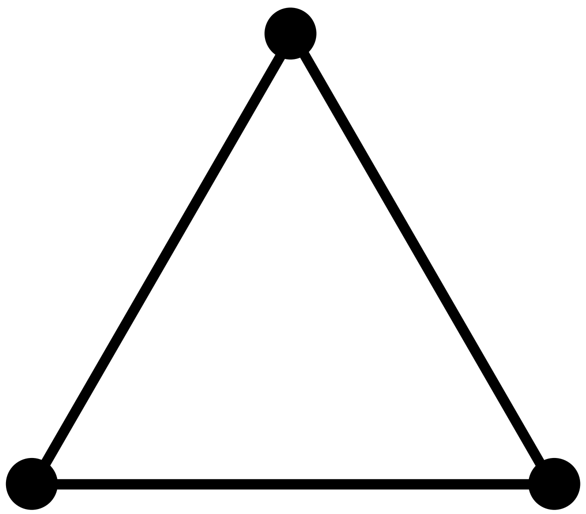
based on joint work with A. Abrams, W. Lam

The *chromatic polynomial* $\chi_G(n)$ of a graph G is the number of proper colorings with n colors.

(adjacent vertices have different colors)



$$\chi(n) = n(n-1)(n-2)$$

χ satisfies a contraction-deletion rule:

$$\chi_G(n) = \chi_{G-e}(n) - \chi_{G/e}(n)$$

but is $\#P$-hard to compute in general.

The Dirichlet problem

A graph $G = (V, E)$

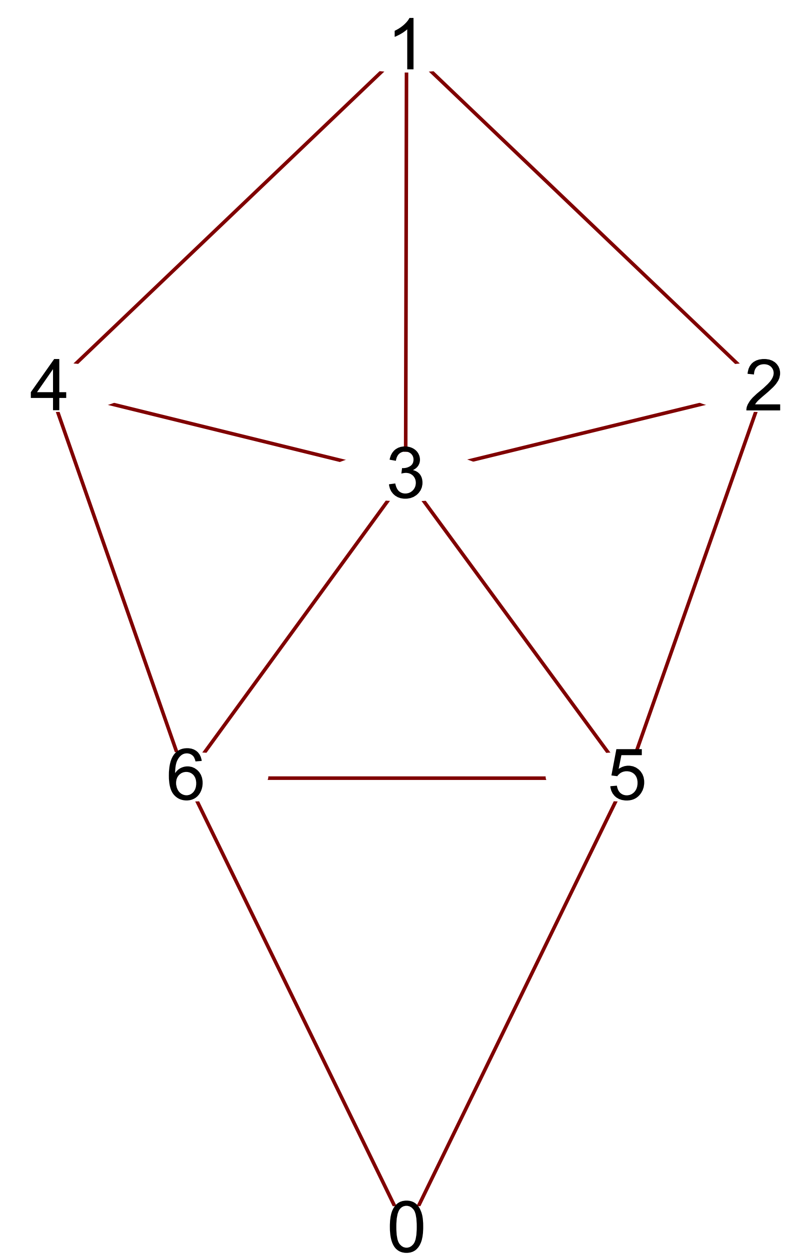
$c : E \to \mathbb{R}_{>0}$ the edge conductances

$B \subset V$ boundary vertices

$u : B \to \mathbb{R}$ boundary values

Find $f : V \to \mathbb{R}$ harmonic on $V \setminus B$

and $f|_B = u$.

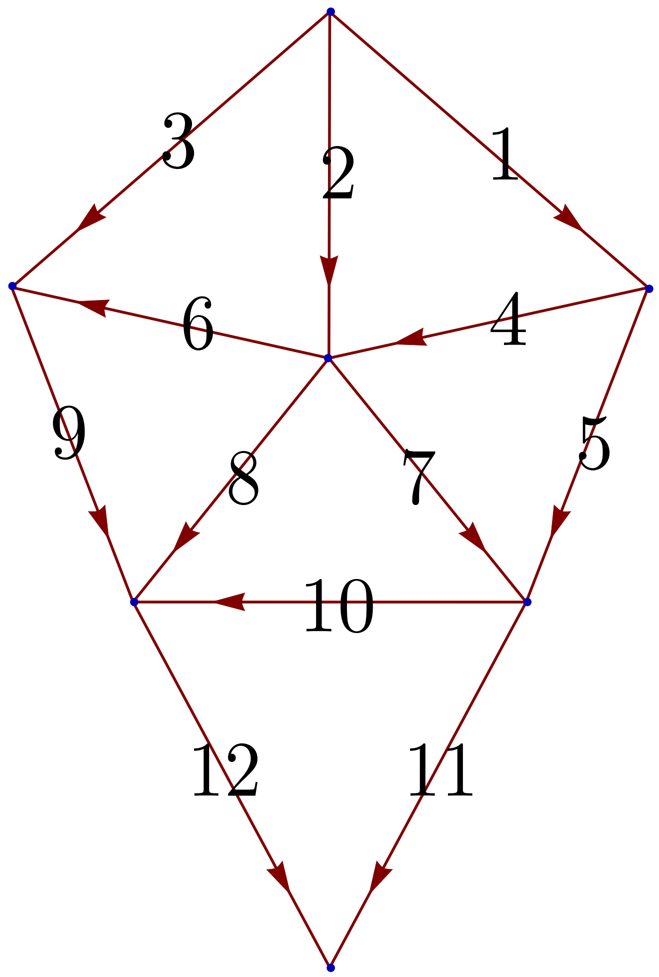


$$0 = \Delta f(x) = \sum_{y \sim x} c_e (f(x) - f(y))$$

f is the unique function with $f|_B = u$ minimizing the Dirichlet energy

$$\mathcal{E}(f) = \sum_{e=xy} \underbrace{c_e (f(x) - f(y))^2}_{\text{edge energy}}$$

A harmonic function induces a *compatible orientation:* an acyclic orientation with no internal sources or sinks, and no oriented paths from lower boundary values to higher boundary values. “current flows downhill”



We let Σ be the set of compatible orientations

How many are there?

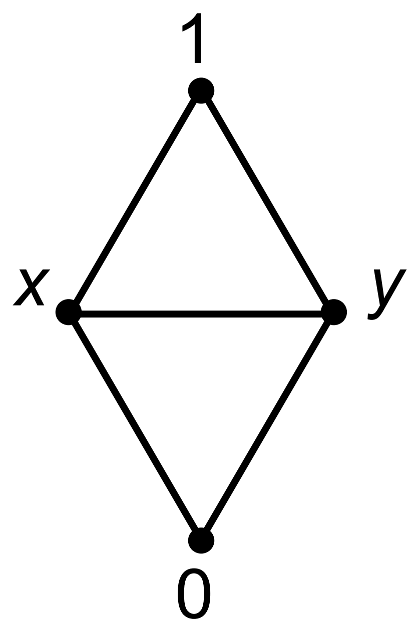
Let $\mathcal{F} \subset \mathbb{R}^V$ be the set of functions with boundary values u and no internal extrema.

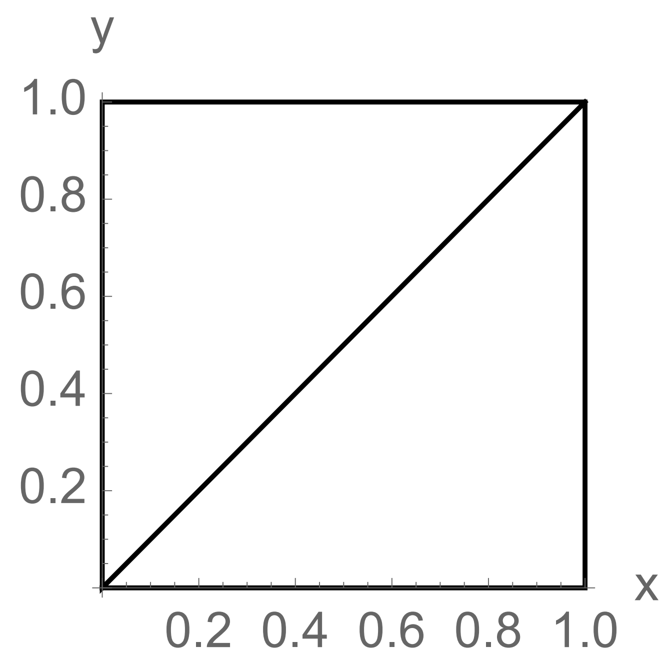
Then

$$\mathcal{F} = \bigcup_{\sigma \in \Sigma} \bar{F}_\sigma$$

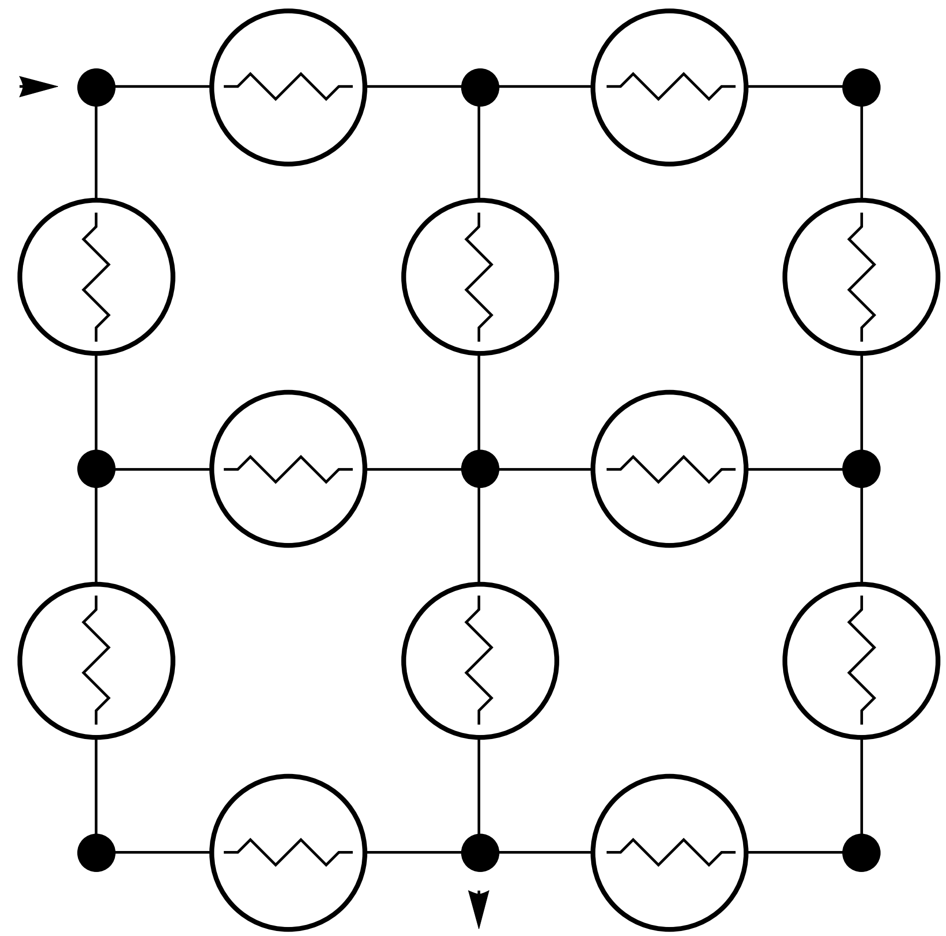
where $\mathcal{F}_\sigma = \{f \in \mathcal{F} \mid \text{sign}(df) = \sigma\}$.

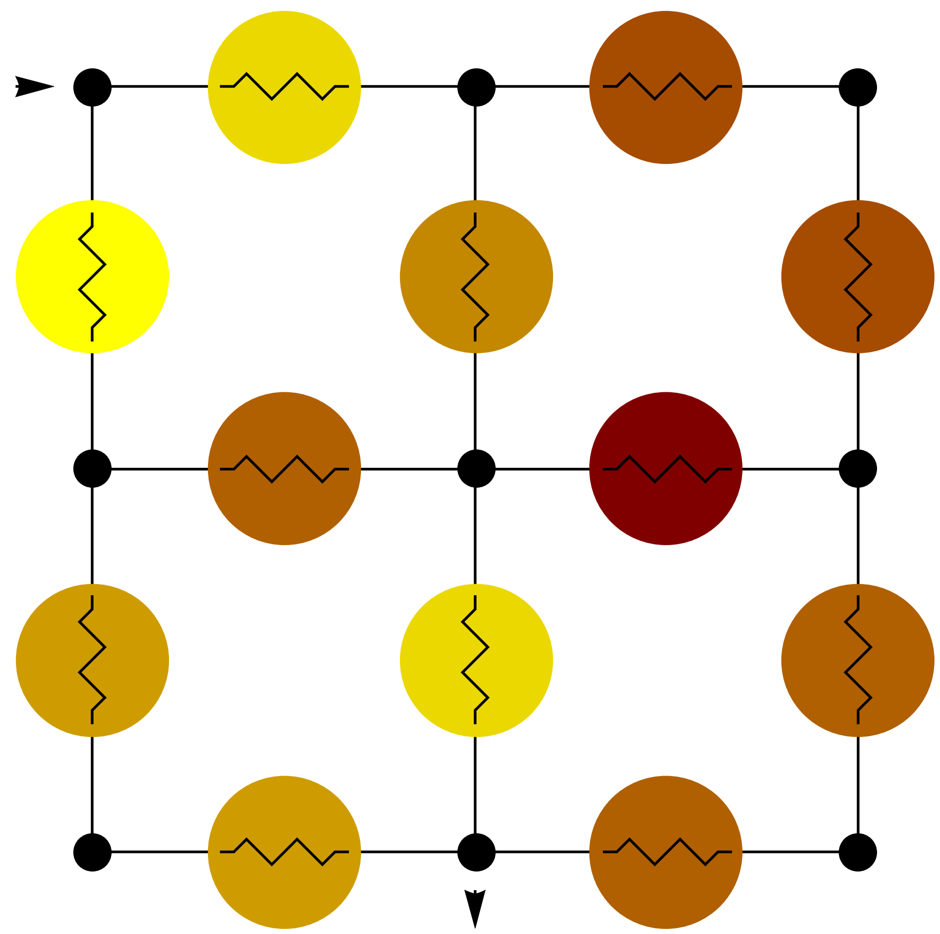
The $\mathcal{F}_\sigma$ are convex polytopes.





Fixed energy problem:

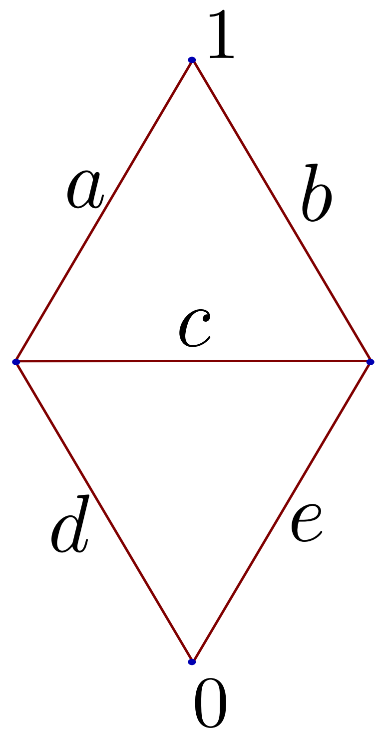




Can we adjust *edge conductances* so that all bulbs burn with the same brightness?

Let $\Psi : (0,\infty)^E \to [0,\infty)^E$ be the map from conductances to energies.

Example



$$\Psi(a,b,c,d,e) = \Big(\frac{a(bd+cd+ce+de)^2}{(ab+ac+ae+bc+bd+cd+ce+de)^2}, \frac{b(ae+cd+ce+de)^2}{(ab+ac+ae+bc+bd+cd+ce+de)^2},$$

$$\frac{c(bd-ae)^2}{(ab+ac+ae+bc+bd+cd+ce+de)^2}, \frac{d(ab+ac+ae+bc)^2}{(ab+ac+ae+bc+bd+cd+ce+de)^2},$$

$$\frac{e(ab+ac+bc+bd)^2}{(ab+ac+ae+bc+bd+cd+ce+de)^2} \Big)$$

Let $\Psi : (0,\infty)^E \to [0,\infty)^E$ be the map from conductances to energies.

Theorem 1: For any $\sigma \in \Sigma$ and $\{\mathcal{E}_e > 0\}$ there is a unique choice of conductances $\{c_e\}$ for which the associated harmonic function realizes this data.

exactly

Thm 2: The harmonic functions h with energies $\{\mathcal{E}_e\}$ are $^\vee$ the solutions to the system of equations

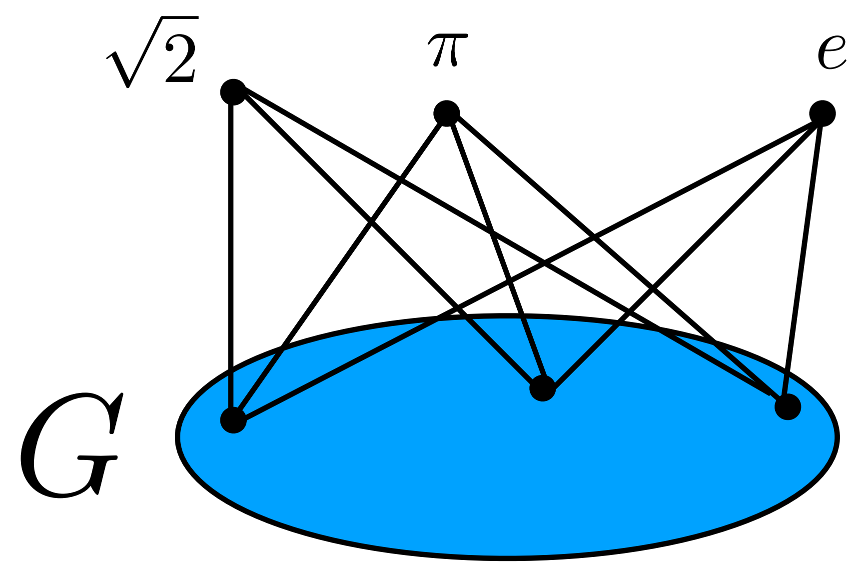
$$\forall x \in V \setminus B, \qquad 0 = \sum_{y\sim x} \frac{\mathcal{E}_e}{h(x)-h(y)}$$

the **enharmonic equation**

"energy – harmonic"

Thm 3: The number of solutions $N = |\Sigma|$ satisfies the contraction-deletion rule $N_G = N_{G-e} + N_{G/e}$.

Cor: Let G_k be obtained from G by adding k vertices as boundary, attached to all vertices of G. Then for any choice of k distinct boundary values, the number of compatible orientations is $\chi_G(1-k)$.



Lemma: The Jacobian determinant of Ψ has the form

$$\det J_\Psi = \prod_{e=xy} (h(x) - h(y))^2.$$

Corollary:

$$|\chi(1-k)| = \frac{1}{|\Delta_m|} \int_{\Delta_m} \prod_{e=xy} \frac{(h(x) - h(y))^2}{Z} \, d\mathrm{vol}$$

where the integral is over the m-simplex Δ_m of conductances summing to 1, and $Z = \sum_{e=xy} c_{xy}(h(x) - h(y))^2$.

Proof of Theorems 1 and 2:

$$0 = \Delta h(x) = \sum_{y \sim x} c_e (h(x) - h(y))$$

recall $\mathcal{E}_{xy} = c_{xy}(h(x) - h(y))^2$

$$= \sum_{y \sim x} \frac{\mathcal{E}_e}{h(x) - h(y)}.$$

(One needs also show that all solutions are real.)

Solutions of the enharmonic equation are critical points of the functional

$$M(h) = \prod_e |h(x) - h(y)|^{\mathcal{E}_e}.$$

Note $\log M(h)$ is strictly concave on each polytope $\mathcal{F}_\sigma = \{h \mid \text{sign}(dh) = \sigma\}$.

□

Proof of Theorem 3:

Recall that $\mathcal{E}_e = c_e(h(x) - h(y))^2 = I_e V_e$ where $I_e = c_e(h(x) - h(y))$ and $V_e = h(x) - h(y)$.

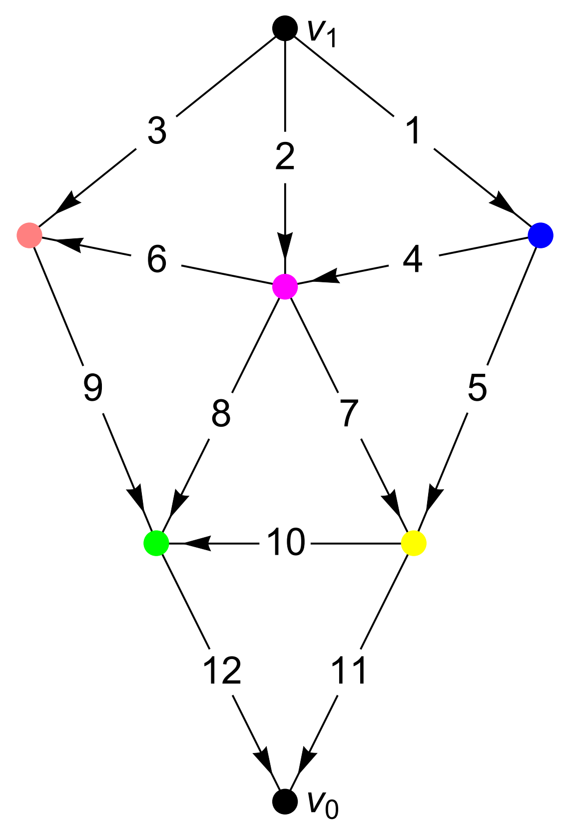
When $\mathcal{E}_e \to 0$, either $I_e \to 0$ or $V_e \to 0$ (or both). In the first case, delete the edge; in the second contract the edge.

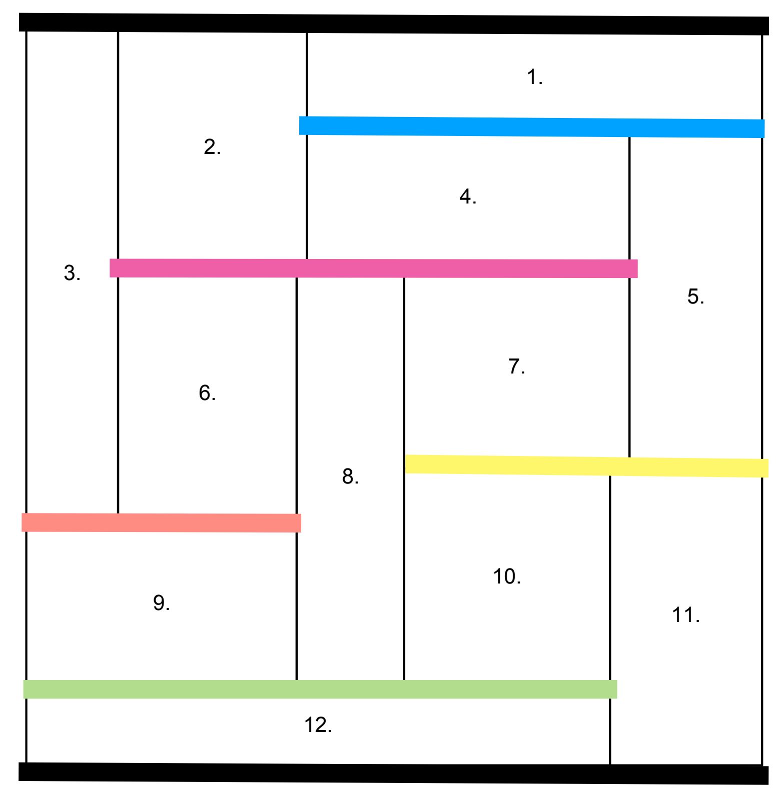
Conversely, the operation of contracting or deleting is reversible by adding in an edge of small energy. □

APPLICATIONS

Smith diagram of a planar network

(with a harmonic function)





voltage = y-coordinate

edge = rectangle

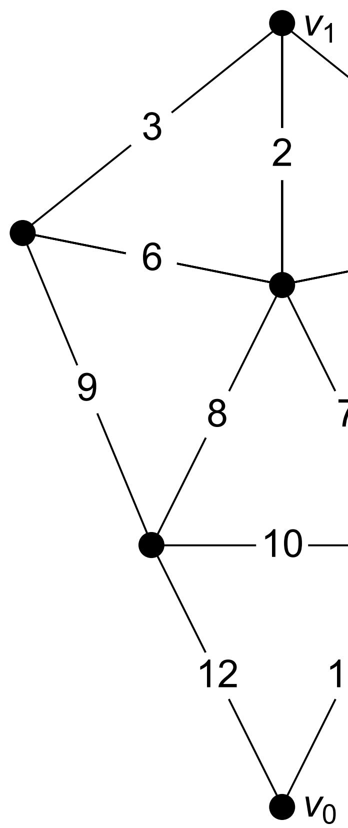
current = width

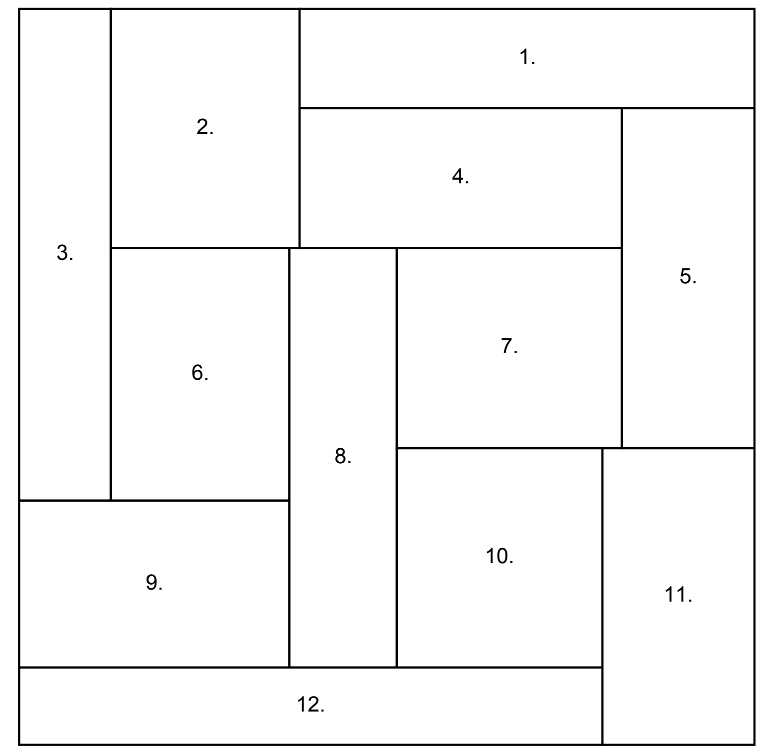
conductance = aspect ratio

energy = area

This graph has 12 acyclic orientations with source at 1 and sink at 0.

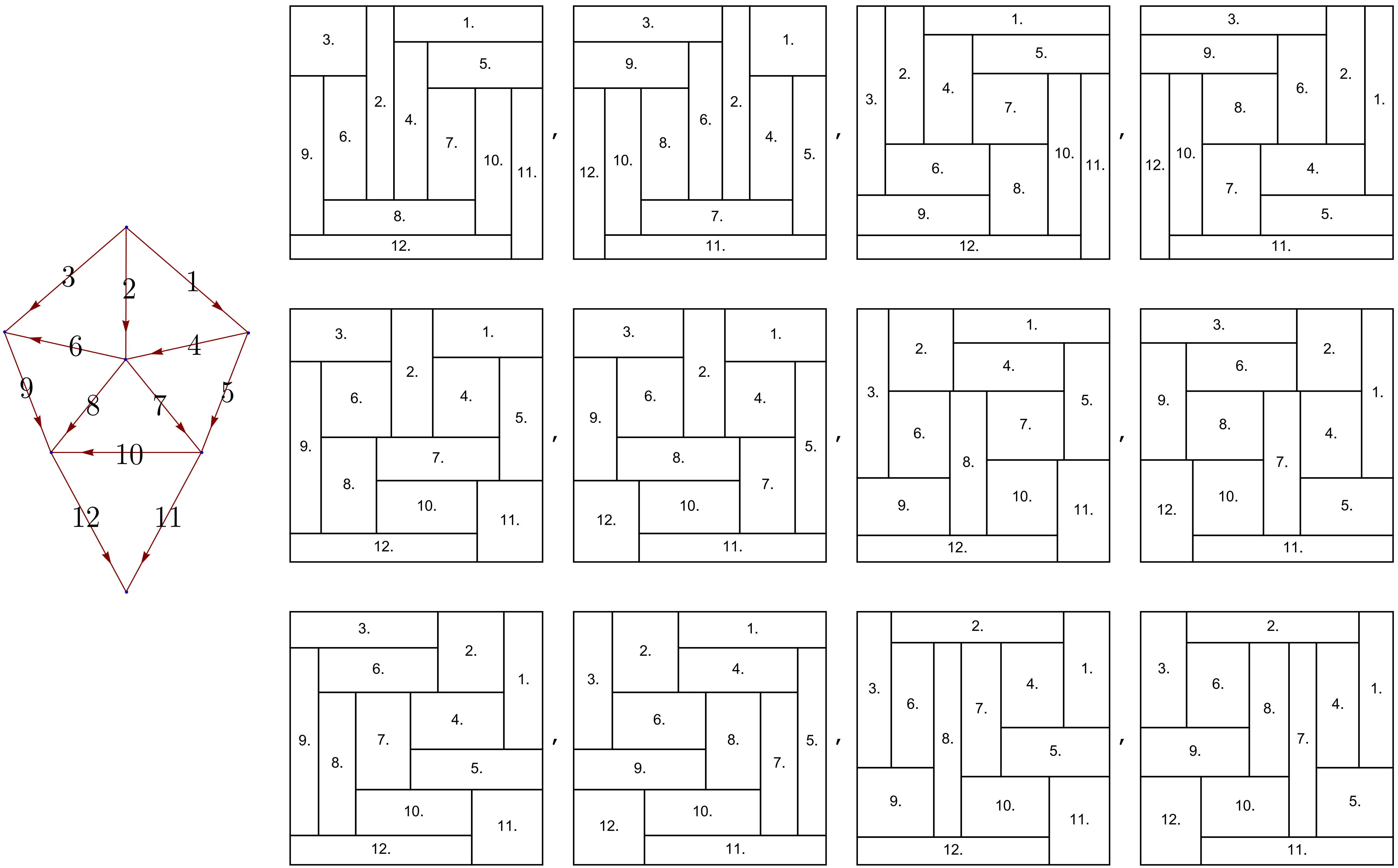
$(|\Sigma| = 12.)$



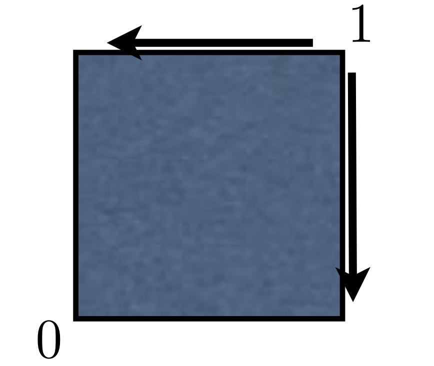


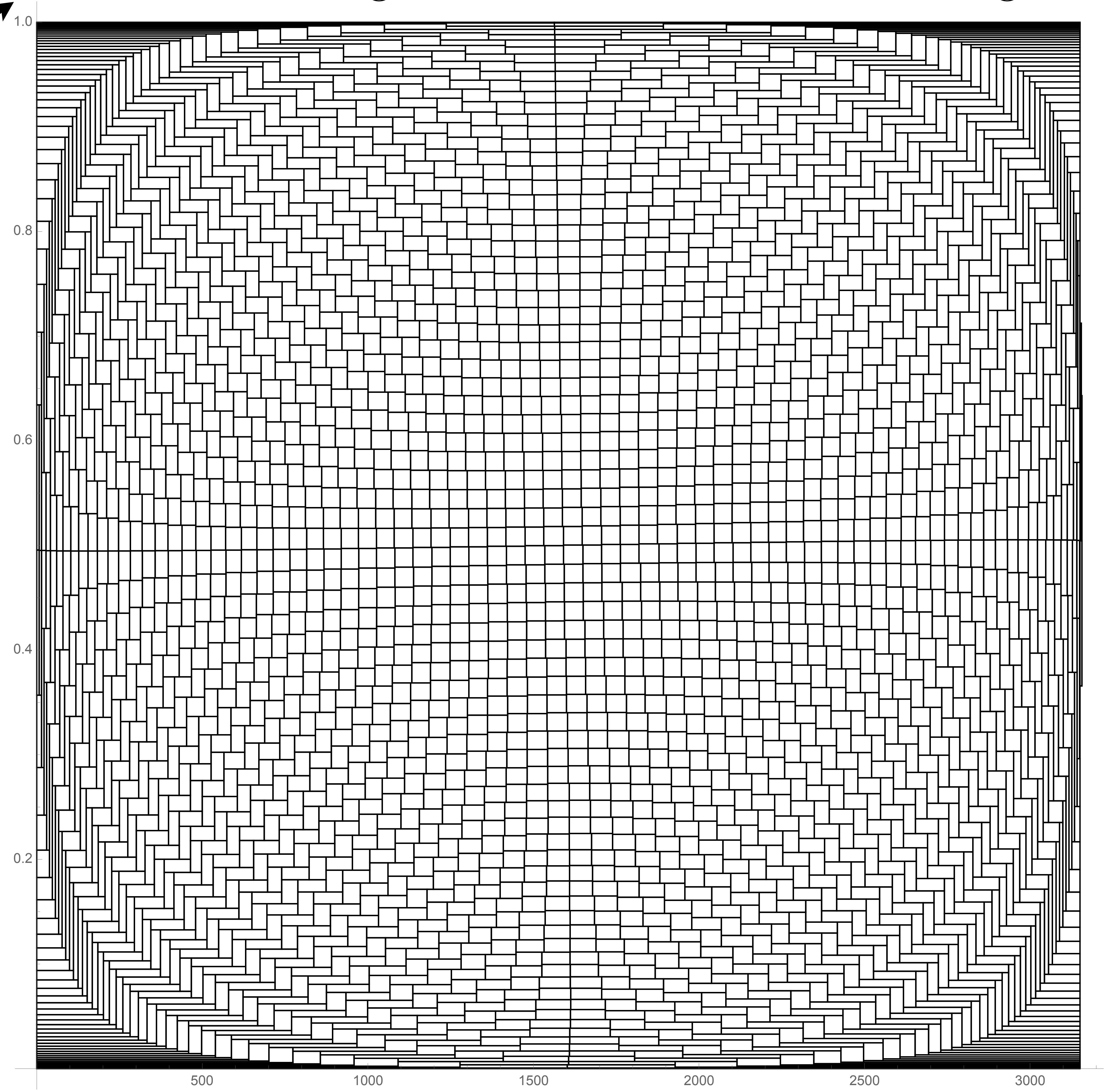
width(1) is the root of a polynomial:

$$2315250000z^{12} - 107438625000z^{11} + 2230924692500z^{10} - 27361273241750z^{9} + 220350695004825z^{8} - 1225394593409700z^{7} + 4817113876088640z^{6} - 13468300499707200z^{5} + 26554002301384704z^{4} - 35985219877131264z^{3} + 31817913970765824z^{2} - 16489700865736704z + 3791571715620864 = 0$$

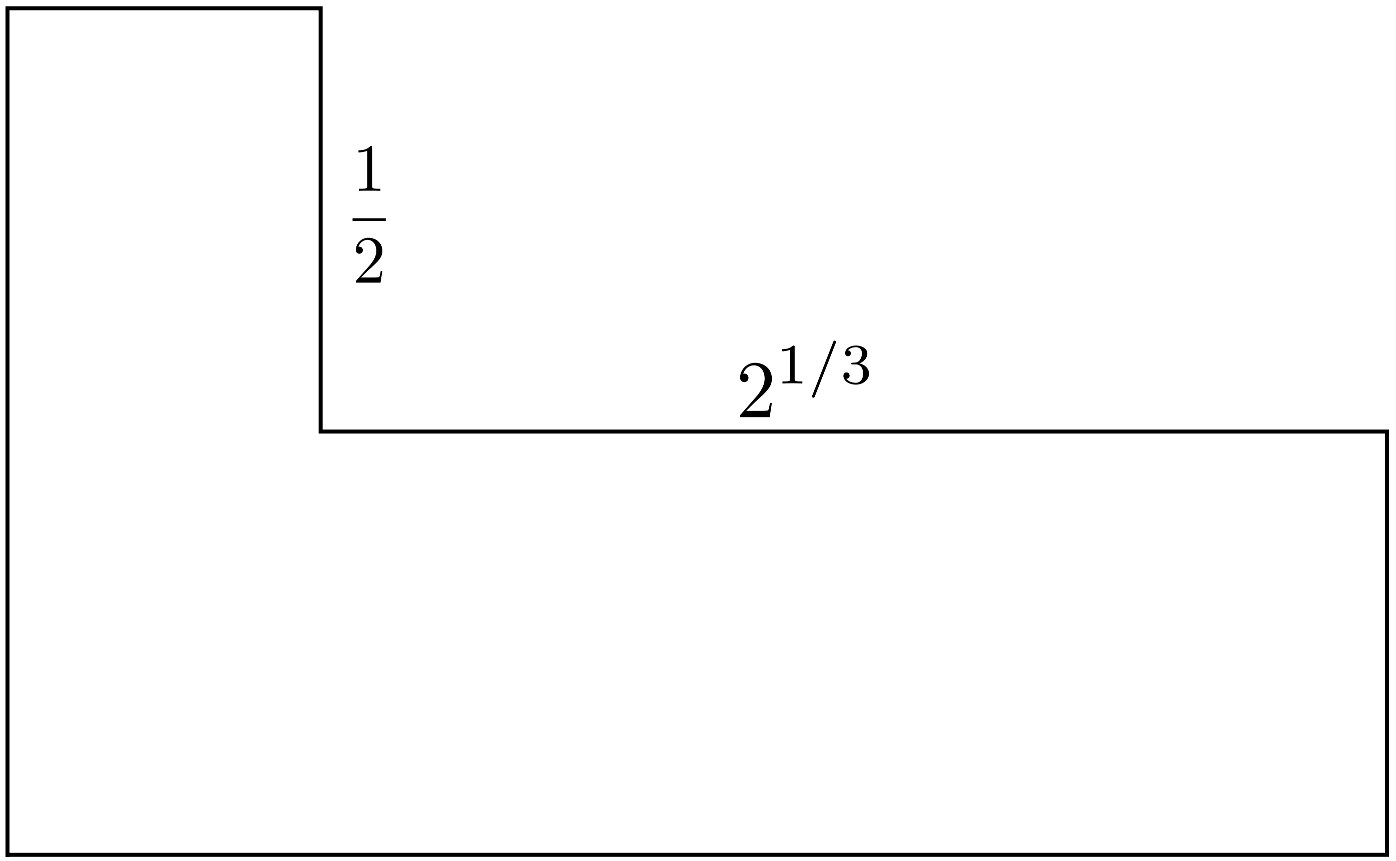

3
2
1
6
4
9
8
7
5
10
12
11

One of the $\approx 10^{599}$ area-1 rectangulations based on the 40×40 grid





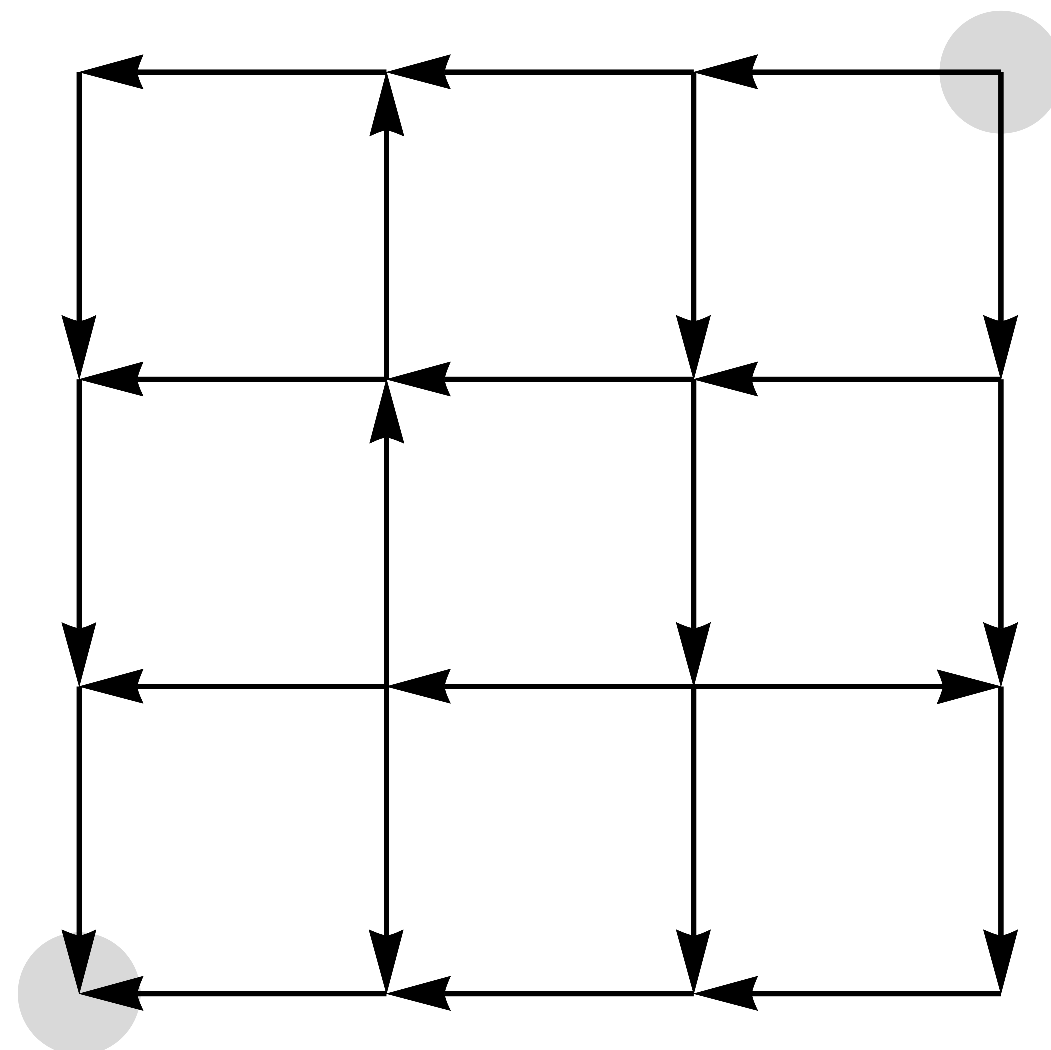
Cor. If a polygon P can be tiled with rectangles of rational area, horizontal lengths are in a totally real extension field of $\mathbb{Q}[v_1, \ldots, v_k]$.

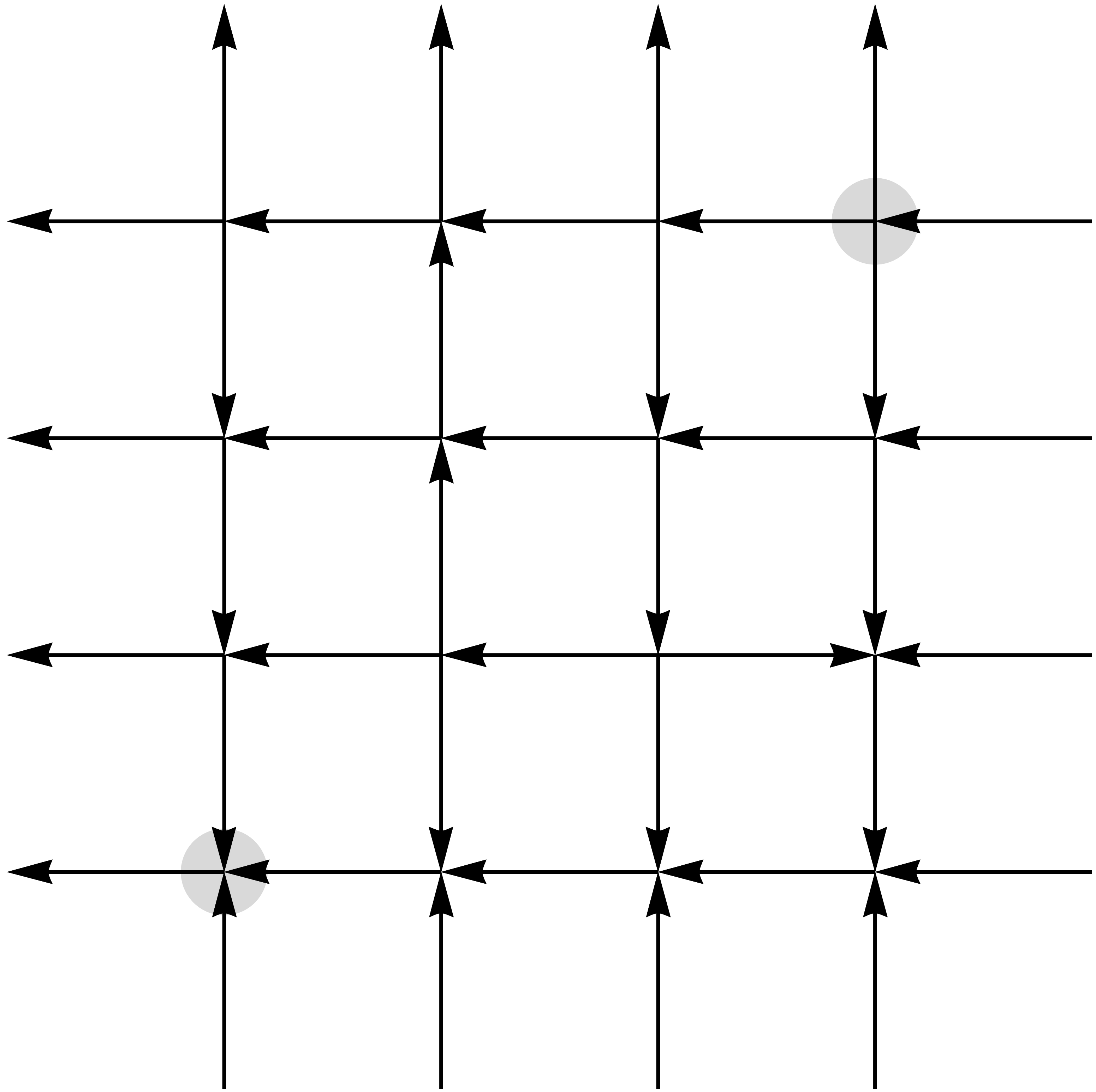


Area 1 but not tileable.

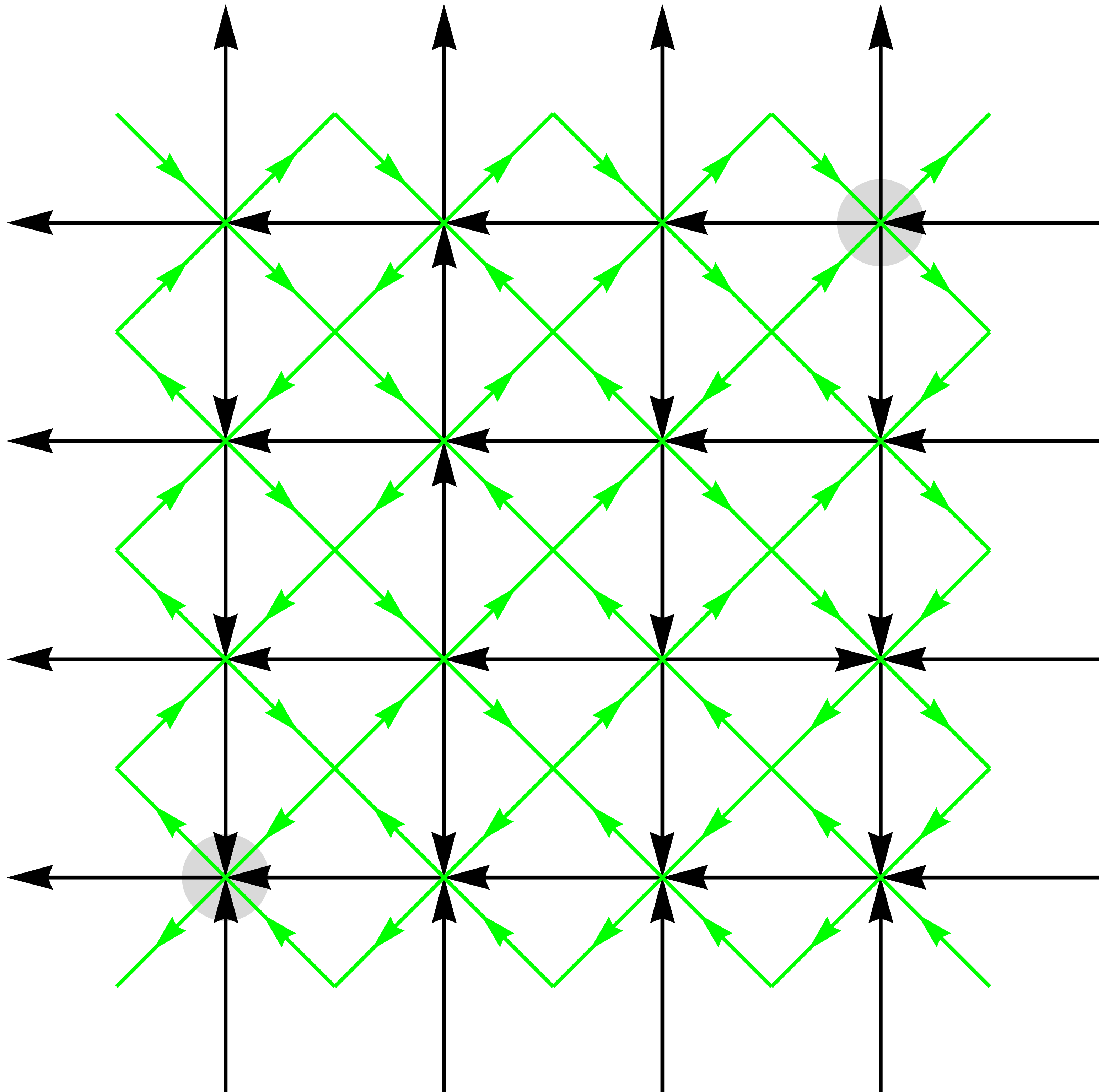
Bipolar orientations on $\mathbb{Z}^2$ and square ice

(with Miller, Sheffield, Wilson)



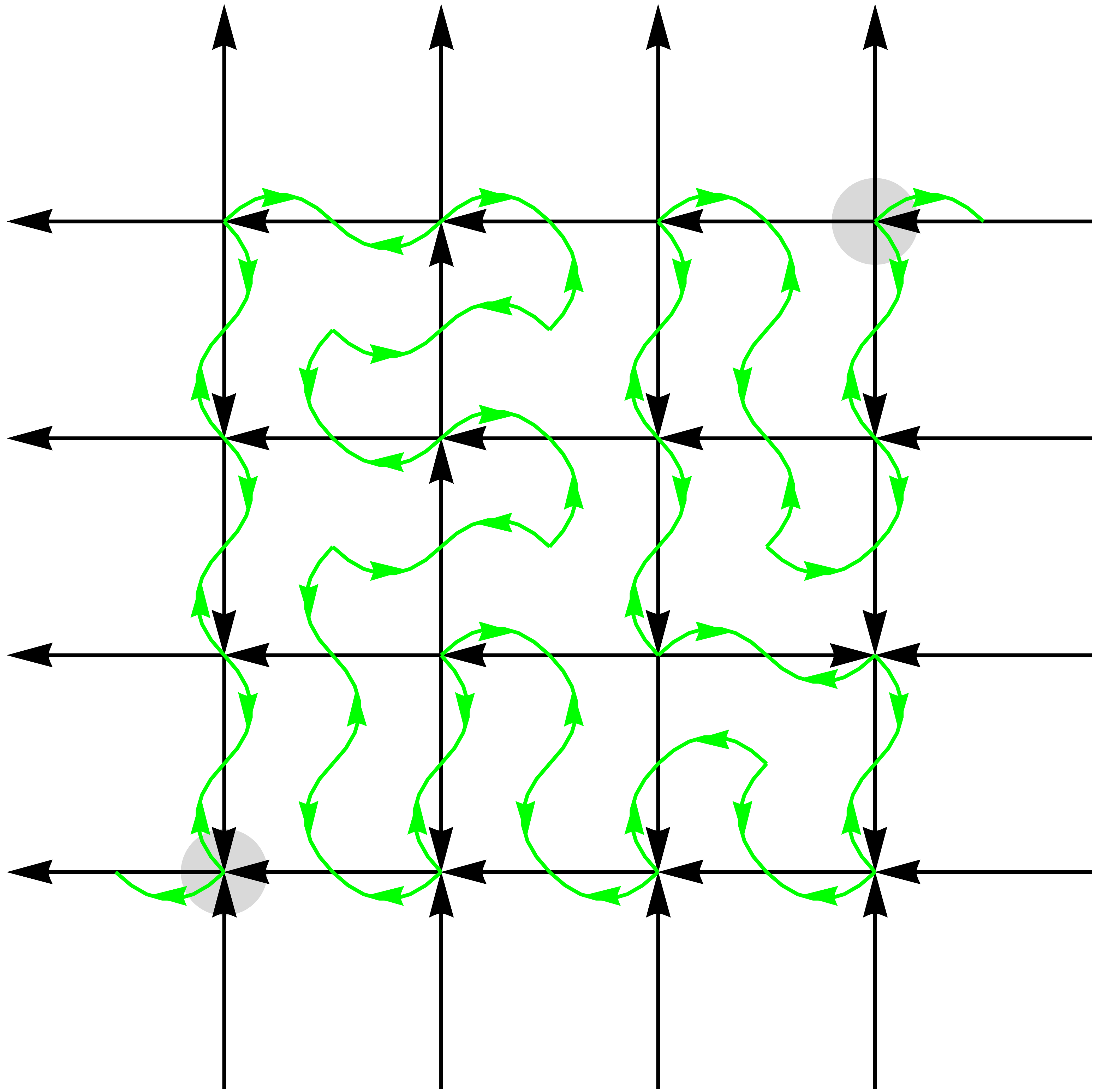


add a row of edges around the boundary oriented N and W

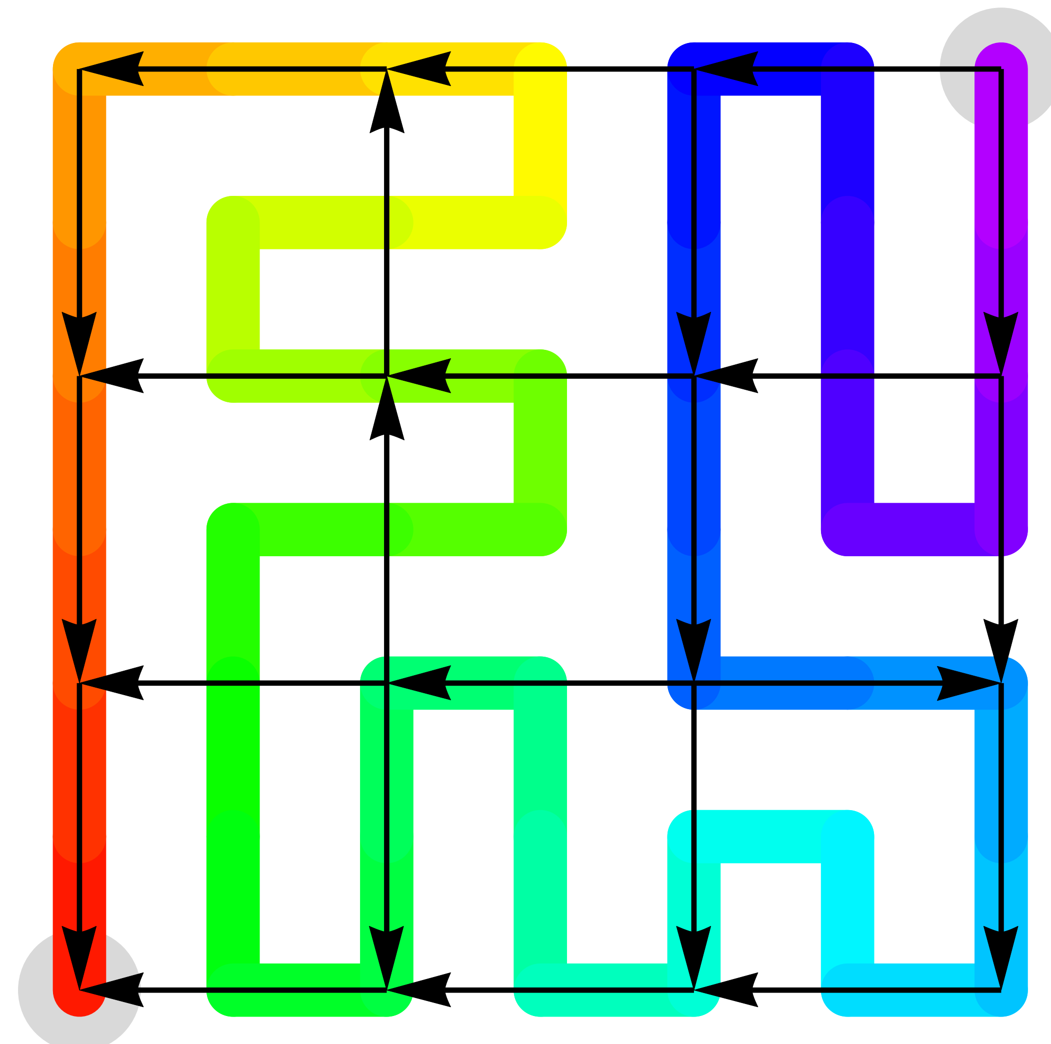


From each black vertex, the two outgoing green arrows separate
the incoming and outgoing black arrows

From each face center, outgoing green arrows point to the face max and min.



Bend outgoing edges right if from vertices, left if from faces.



The associated peano curve, colored according to distance traveled

(Conjectural) SLE_{12} scaling limit

THANK YOU