Binomial irreducible decomposition

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joint with Thomas Kahle

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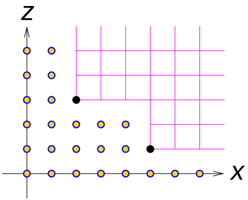
Background text

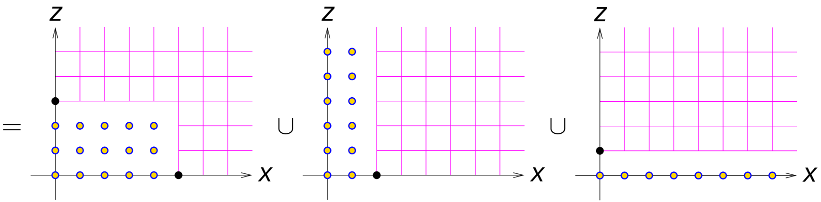
Monomial irreducible decomposition combinatorially expresses the (exponents on the) monomials outside of a monomial ideal as a union of box-shaped sets of lattice points. Binomial irreducible decomposition aims for a similar outcome when the input is a binomial ideal, but its existence has until now remained open (Problem 7.5 in the paper by Eisenbud and Sturmfels, listed below). This talk is about equivalence relations and partial orders on commutative monoids, explicitly described in terms of lattice points, as in the monomial case. The resulting combinatorics, along with a modicum of abelian group character theory, yields binomial irreducible decomposition. This talk covers material in reference 5, below. This work is motivated by and based on the other references. The figures here go with the scanned PDF lecture notes.

1. A. Dickenstein, L. Matusevich, and E. Miller, *Combinatorics of binomial primary decomposition*, Math. Zeitschrift **264**, no. 4 (2010), 745–763.
2. D. Eisenbud and B. Sturmfels, *Binomial ideals*, Duke Math. J. **84** (1996), no. 1, 1–45.
3. R. Gilmer, *Commutative semigroup rings*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1984.
4. P. A. Grillet, *Commutative semigroups*, Advances in Mathematics, Kluwer Academic Publishers, London, 2001.
5. T. Kahle and E. Miller, *Decompositions of commutative monoid congruences and binomial ideals*, 63 pages. `arXiv:math.AC/1107.4699`

Monomial ideals

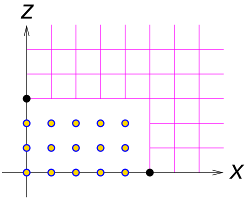


std $\langle x^2 z^3, x^5 z \rangle$

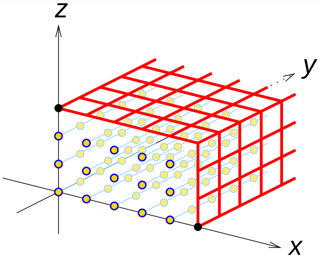


std $\langle x^5, z^3 \rangle$ std $\langle x^2 \rangle$ std $\langle z \rangle$

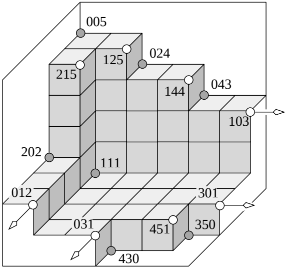
3-dimensional monomial ideals

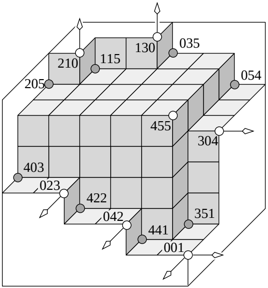


$\langle x^5, z^3 \rangle \subseteq \Bbbk[x, y]$

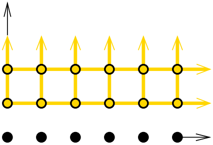


$\langle x^5, z^3 \rangle \subseteq \Bbbk[x, y, z]$

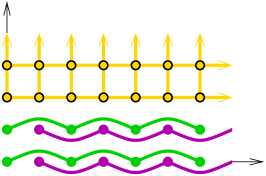




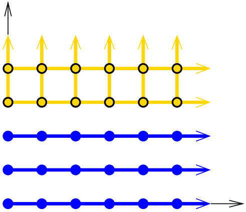
Some binomial ideals



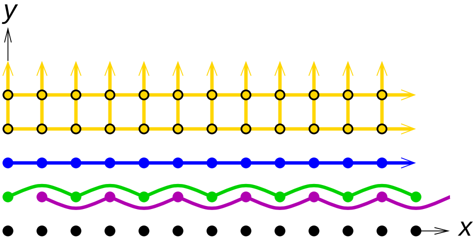
$\langle y \rangle$



$\langle 1 - x^2, y^2 \rangle$

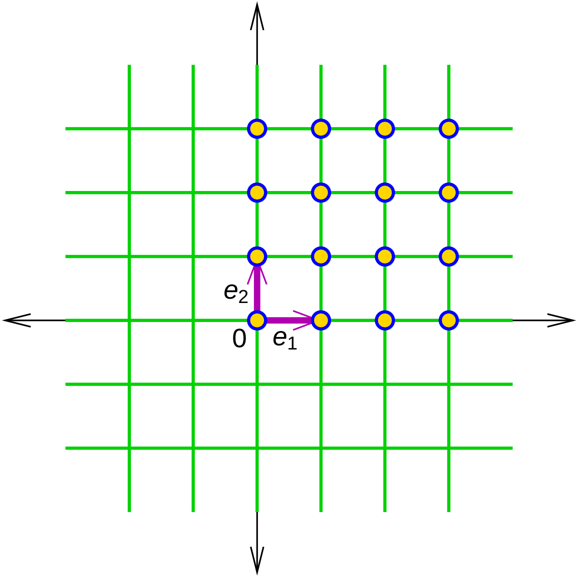


$\langle 1 - x, y^3 \rangle$

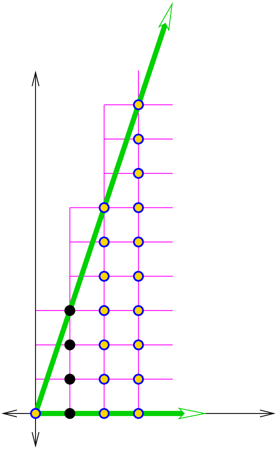


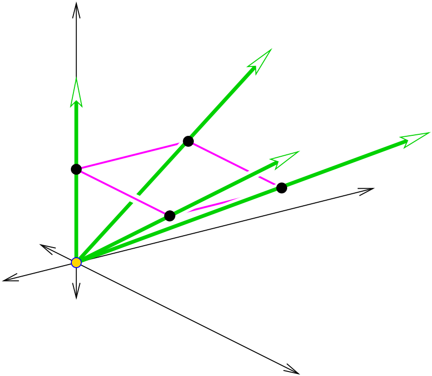
$\langle y - x^2 y, y^2 - x y^2, y^3 \rangle$

$\mathbb{N}^2$

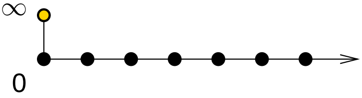


Affine semigroups

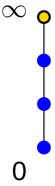




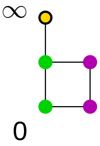
Weirder monoids



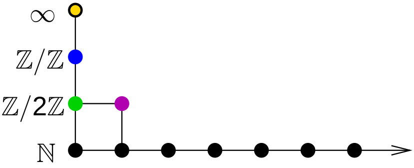
$\mathbb{N} \cup \infty$



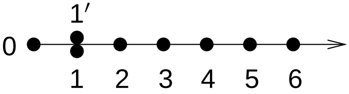
$\mathbb{N}/(3+\mathbb{N})$



$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{N})/(2+\mathbb{N})$

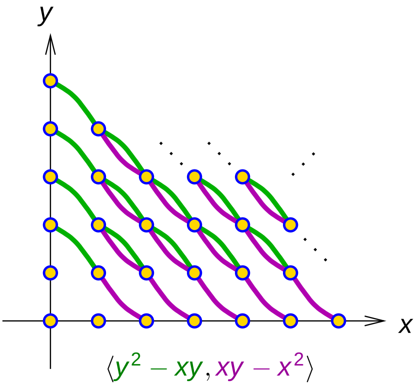


a weird monoid

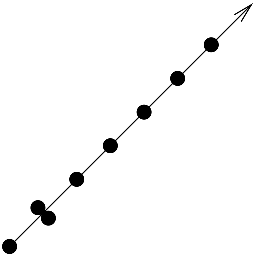


$\mathbb{N}$ with 1 doubled

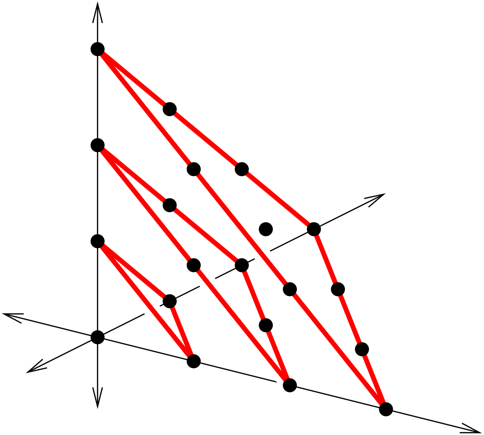
Presenting $\mathbb{N}$ with 1 doubled



$\longrightarrow$



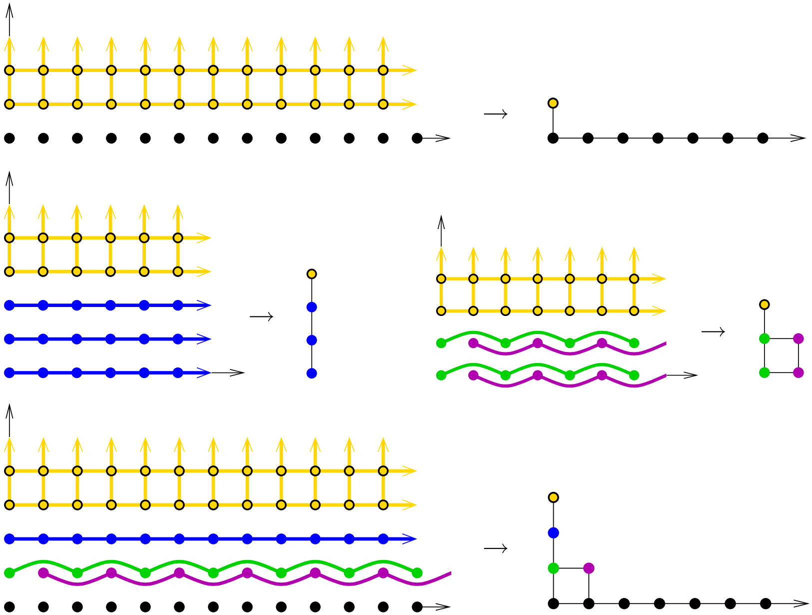
Presenting affine semigroups



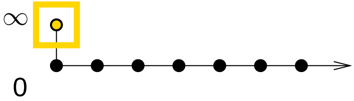
fibers of $\mathbb{N}^3 \to \mathbb{N}$

$(a, b, c) \mapsto a + b + c$

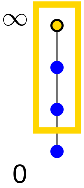
Presenting weird monoids



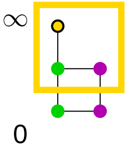
Nilpotents



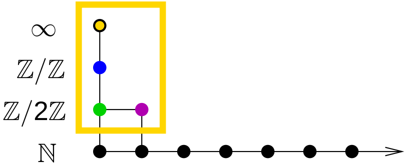
$\mathbb{N} \cup \infty$



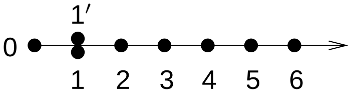
$\mathbb{N}/(3 + \mathbb{N})$



$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{N})/(2 + \mathbb{N})$

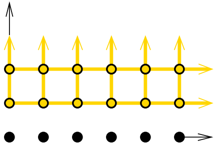


a weird monoid

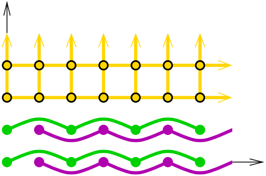


$\mathbb{N}$ with 1 doubled

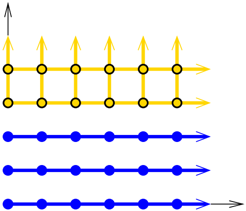
Coprincipal decomposition



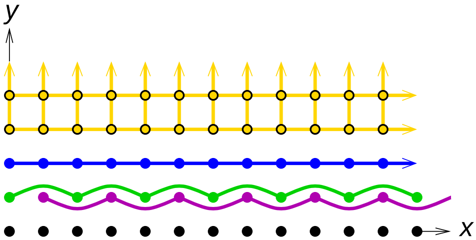
$\langle y \rangle$



$\langle 1 - x^2, y^2 \rangle$



$\langle 1 - x, y^3 \rangle$



$\langle y - x^2 y, y^2 - x y^2, y^3 \rangle$