Algebraic-Geometric ideas in Discrete Optimization

Jesús A. De Loera, UC Davis

new results on several papers joint work with (subsets of):

M. Köppe & J. Lee (IBM),
U. Rothblum & S. Onn (Technion Haifa),
R. Hemmecke (T.Univ. Munich) & R. Weismantel (ETH Zürich)

November 5, 2011

Le Menu

- Appetizer: Challenges in Discrete Optimization and why the need for new tools...
- Main Dish: Some Algebraic-Geometric Algorithms in Optimization
  - Barvinok's Algorithm.
  - Graver Bases.
- Dessert: Closing Comments and Future directions.

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Challenges in Discrete Optimization why need for new tools

(in particular from algebra, geometry and topology).

What is Discrete Optimization?

- A part of Applied Mathematics, its main problem: Given a finite set X, each of whose elements has an assigned cost, price or optimality criteria, find the cheapest such object.
- Problems come from bioinformatics, industrial engineering, management, operations planning, finances, any area where the best solution is required!
- History starts with WWII Initial work by Kantorovich (1939), T.C Koopmans (1941), von Neumann (1947), Dantzig (1950), Ford and Fulkerson (1956). Invention of linear programming and the simplex method.

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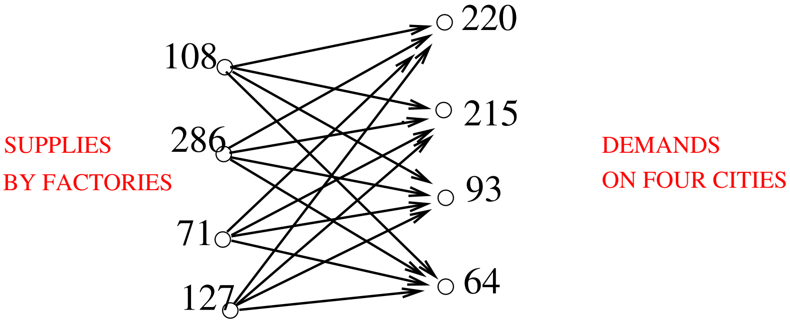
A Useful Example

- The Transportation problem: A company builds laptops in four factories, each with certain supply power. Four cities have laptop demands. There is a cost $c_{i,j}$ for transporting a laptop from factory i to city j. What is the best assignment of transport in order to minimize the cost?



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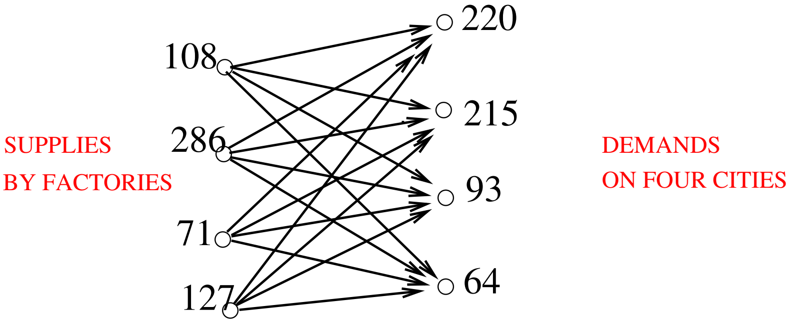

ILP model: equations and inequalities

- Let $x_{i,j}$ be a variable indicating number of laptops factory i provides to city j. $x_{i,j}$ can only take non-negative integer values, $x_{i,j} \geq 0$.
- Then Since factory i produces a_i laptops we have

$$\sum_{j=1}^{n} x_{i,j} = a_i, \text{ for all } i = 1, \ldots, n.$$

and since city j needs b_j laptops

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- Now we minimize $\sum c_{i,j} x_{i,j}$.

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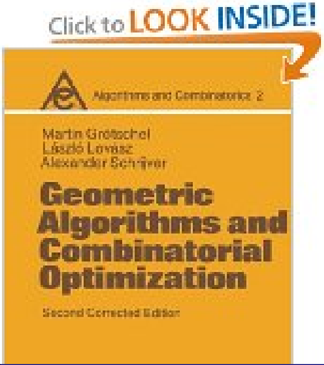
Overview LINEAR Discrete Optimization



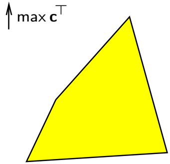
Efficient computation with Convex Sets & Lattices $\Longleftrightarrow$ Efficient Optimization



At the beginning there was...

Linear programs

$$\begin{array}{ll} \max & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \le \mathbf{b} \end{array}$$



Special integer programs

$$\begin{array}{ll} \max & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \text{all } x_i \text{ integer} \end{array}$$

Matrix A is SPECIAL!

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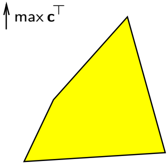

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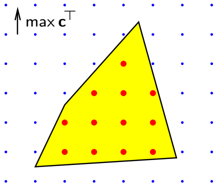

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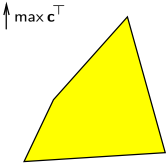

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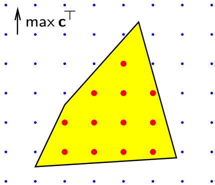

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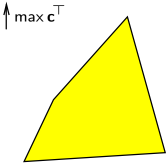

Easy
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Medium
(can be easy or hard)
Network problems
Fixed dimension
knapsacks
0-1 matrices

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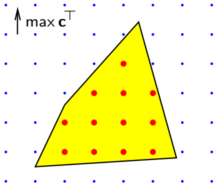

Hard
(NP-hard)

Traditional Algorithms

Dual (polyhedral) techniques

Cutting plane algorithms
– based on polyhedral theory

Enumeration

Branch-and-bound

Adhoc methods

special structure
(e.g. network,
matroids, etc.)

Mathematical modelling – Strong initial IP formulation

Traditional Algorithms

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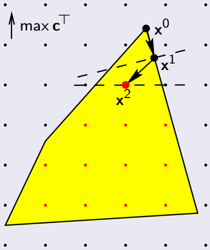

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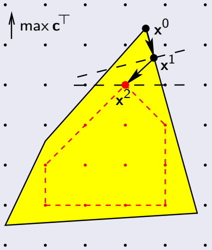

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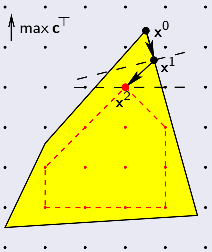

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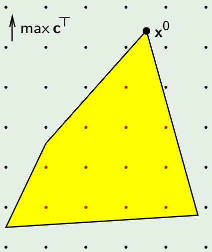

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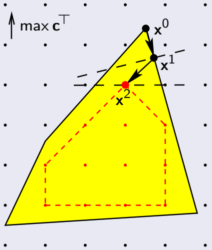

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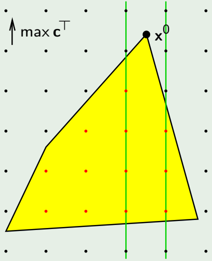

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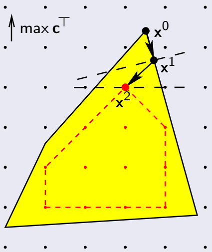

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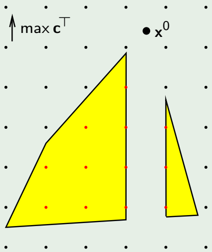

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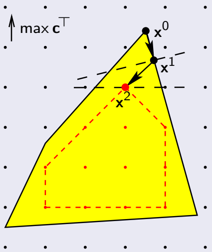

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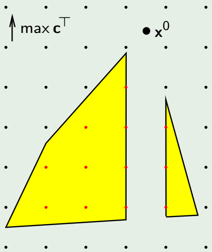

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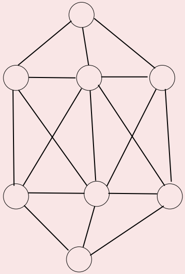

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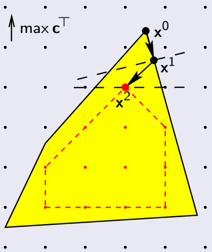

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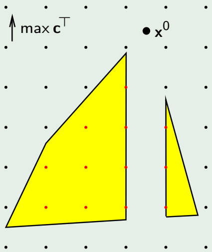

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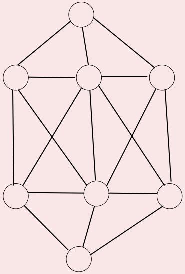

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Mathematical modelling – Strong initial IP formulation

OUR WISH:

Want to handle more complicated Constraints and Objective functions

Example: Non-linear transportation polytopes

1. In the traditional transportation problem cost at an edge is a constant. Thus we optimize a linear function.
2. but due to congestion or heavy traffic or heavy communication load the transportation cost on an edge could be a non-linear function of the flow at each edge.
3. For example cost at each edge is $f_{ij}(x_{ij}) = c_{ij}|x_{ij}|^{a_{ij}}$ for suitable constant a_{ij}. This results on a non-linear function $\sum f_{ij}$ which is much harder to minimize.

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Reality is NON-LINEAR and worse!!

Non-linear Discrete Optimization

$$\max/\min \ f(x_1,\ldots,x_d)$$
$$\text{subject to } g_j(x_1,\ldots,x_d) \leq 0,$$

for $j = 1\ldots s$, and with

with x_i integer

with f, g_j Non-Linear

WHAT CAN BE DONE IN THIS GENERAL CONTEXT??

Prove good theorems? Are there efficient algorithms?

- BAD NEWS: The problem is INCREDIBLY HARD

  Theorem It is UNDECIDABLE already when f, g_i's are arbitrary polynomials (Jeroslow, 1979).

- EVEN WORSE

  Theorem: It undecidable even with number of variables=10. (Matiyasevich and Davis 1982).

- THERE IS HOPE with good structure:

  Theorem: For fixed number of variables AND convex polynomials f, g_i problem can be solved in polynomial time

  (Khachiyan and Porkolab, 2000)

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Let f be a multivariate polynomial function,

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Special programs

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Special programs

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We study TWO
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Algebraic Geometric Ideas in Optimization

Problem type

$$\begin{array}{rl} \max & f(x_1,\ldots,x_d) \\ \text{subject to} & (x_1,\ldots,x_d) \in P \cap \mathbf{Z}^d, \end{array}$$

where

- P is a polytope (bounded polyhedron) given by linear constraints,
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- the dimension d is fixed.

Prior Work

- Integer Linear Programming can be solved in polynomial time
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Applications of Barvinok's Algorithms

Idea: New Representation of Lattice Points

- Given $K \subset \mathbf{R}^d$ we define the formal power series

$$f(K) = \sum_{\alpha \in K \cap \mathbf{Z}^d} z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}.$$

Think of the lattice points as monomials!!! EXAMPLE: $(7, 4, -3)$ is $z_1^7 z_2^4 z_3^{-3}$.

- **Theorem** (see R. Stanley EC Vol 1) Given $K = \{x \in \mathbf{R}^n | Ax = b,\ Bx \leq b'\}$ where A, B are integral matrices and b, b' are integral vectors, The generating function $f(K)$ can be encoded as rational function.

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Barvinok's short rational generating functions

Generating functions

$$g_P(z) = z^0 + z^1 + z^2 + z^3 + \dots z^M$$

Theorem (Alexander Barvinok, 1994)

Let the dimension d be fixed. There is a polynomial-time algorithm for computing a representation of the generating function

$$g_P(z_1, \dots, z_d) = \sum_{(\alpha_1,\dots,\alpha_d)\in P\cap \mathbf{Z}^d} z_1^{\alpha_1} \cdots z_d^{\alpha_d} = \sum_{\alpha \in P\cap \mathbf{Z}^d} \mathbf{z}^\alpha$$

of the integer points $P \cap \mathbf{Z}^d$ of a polyhedron $P \subset \mathbf{R}^d$ (given by rational inequalities) in the form of a rational function.

Corollary

In particular,

$$N = |P \cap \mathbf{Z}^d| = g_P(\mathbf{1})$$

can be computed in polynomial time (in fixed dimension).

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$$g_P(z) = z^0 + z^1 + z^2 + z^3 + \dots z^M$$
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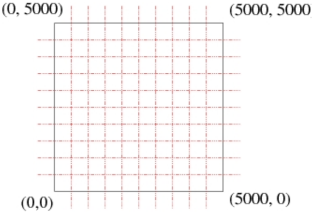
Example

Let P be the square with vertices $V_1 = (0,0)$, $V_2 = (5000,0)$, $V_3 = (5000,5000)$, and $V_4 = (0,5000)$.



The generating function $f(P)$ has over 25,000,000 monomials,
$f(P) = 1 + z_1 + z_2 + z_1^1 z_2^2 + z_1^2 z_2 + \cdots + z_1^{5000} z_2^{5000}$,

But it can be written using only four rational functions

$$\frac{1}{(1-z_1)(1-z_2)} + \frac{{z_1}^{5000}}{(1-{z_1}^{-1})(1-z_2)} + \frac{{z_2}^{5000}}{(1-{z_2}^{-1})(1-z_1)} + \frac{{z_1}^{5000}{z_2}^{5000}}{(1-{z_1}^{-1})(1-{z_2}^{-1})}$$

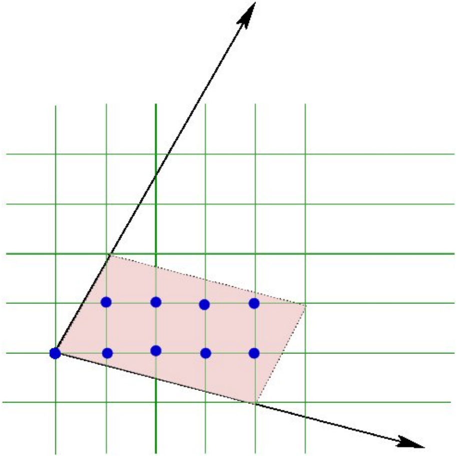
Also, $f(tP, z)$ is

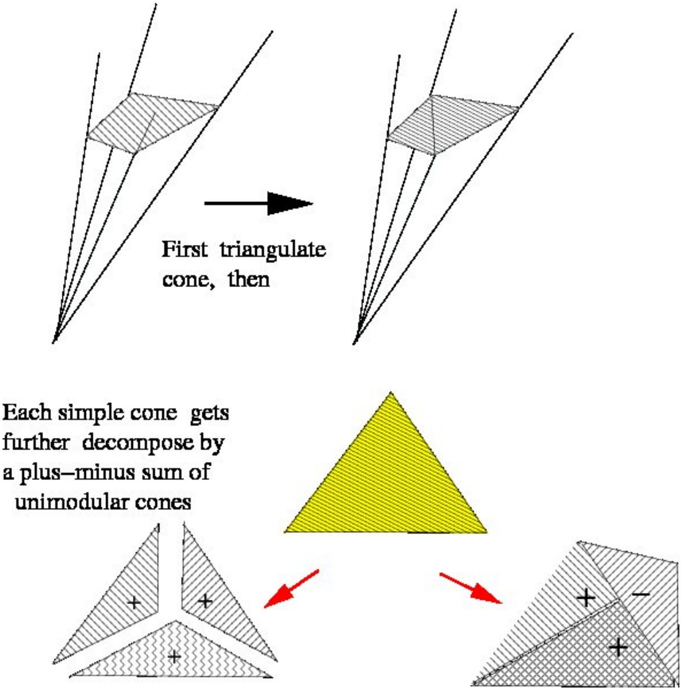
$$\frac{1}{(1-z_1)(1-z_2)} + \frac{{z_1}^{5000\cdot t}}{(1-{z_1}^{-1})(1-z_2)} + \frac{{z_2}^{5000\cdot t}}{(1-{z_2}^{-1})(1-z_1)} + \frac{{z_1}^{5000\cdot t}{z_2}^{5000\cdot t}}{(1-{z_1}^{-1})(1-{z_2}^{-1})}$$

Rational Function of a pointed Cone

EXAMPLE: we have $d = 2$ and $c_1 = (1, 2)$, $c_2 = (4, -1)$. We have:

$$f(K) = \frac{z_1^4 z_2 + z_1^3 z_2 + z_1^2 z_2 + z_1 z_2 + z_1^4 + z_1^3 + z_1^2 + z_1 + 1}{(1 - z_1 z_2^2)(1 - z_1^4 z_2^{-1})}.$$




First triangulate
cone, then
Each simple cone gets
further decompose by
a plus–minus sum of
unimodular cones

Theorem (FPTAS for Integer Polynomial Maximization)

Let the dimension d be fixed. There exists an algorithm whose input data are

- a polytope $P \subset \mathbf{R}^d$, given by rational linear inequalities, and
- a polynomial $f \in \mathbf{Z}[x_1, \dots, x_d]$ with integer coefficients and maximum total degree D that is non-negative on $P \cap \mathbf{Z}^d$

with the following properties.

1. For a given k, it computes in running time polynomial in k, the encoding size of P and f, and D lower and upper bounds $L_k \leq f(\mathbf{x}^{\max}) \leq U_k$ satisfying
$$U_k - L_k \leq \left(\sqrt[k]{|P \cap \mathbf{Z}^d|} - 1 \right) \cdot f(\mathbf{x}^{\max}).$$
2. For $k = (1 + 1/\epsilon) \log(|P \cap \mathbf{Z}^d|)$, the bounds satisfy
$$U_k - L_k \leq \epsilon f(\mathbf{x}^{\max}),$$
and they can be computed in time polynomial in the input size, the total degree D, and $1/\epsilon$.
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Differential operators on generating functions

The Euler differential operator $\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)$ maps:

$$g(z) = \sum_{j=0}^{D} g_j z^j \quad \longmapsto \quad z\frac{\mathrm{d}}{\mathrm{d}z} g(z) = \sum_{j=0}^{D} (j \cdot g_j) z^j$$

$$g_P(z) = z^0 + z^1 + z^2 + z^3 + z^4$$

Apply differential operator:

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right) g_P(z) = 1z^1 + 2z^2 + 3z^3 + 4z^4$$

Apply differential operator again:

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Lemma

$$f(x_1,\dots,x_d)=\sum_\beta c_\beta \mathbf{x}^\beta \in \mathbf{Z}[x_1,\dots,x_d]$$

can be converted to a differential operator

$$D_f = f\left(z_1\frac{\partial}{\partial z_1},\dots,z_d\frac{\partial}{\partial z_d}\right)=\sum_\beta c_\beta \left(z_1\frac{\partial}{\partial z_1}\right)^{\beta_1}\cdots\left(z_d\frac{\partial}{\partial z_d}\right)^{\beta_d}$$

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Let $g_P(\mathbf{z})$ be the Barvinok generating function of the lattice points of P. Let f be a polynomial in $\mathbf{Z}[x_1,\dots,x_d]$ of maximum total degree D.
We can compute, in polynomial time in D and the size of the input data, a Barvinok rational function representation $g_{P,f}(\mathbf{z})$ for $\sum_{\alpha\in P\cap\mathbf{Z}^d} f(\alpha)\mathbf{z}^\alpha$.

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- We are interested on optimization of a convex function over $\{x \in \mathbf{Z}^n : Ax = b,\ x \geq 0\}$. We will use basic Algebraic Geometry.
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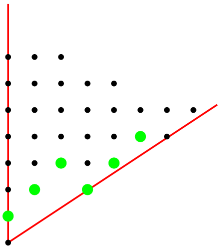
  Eg: $(3, -2, -8, 0, 8) \sqsubset (4, -3, -9, 0, 9)$, incomparable to $(-4, -3, 9, 1, -8)$.



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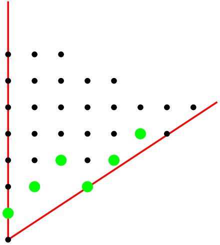

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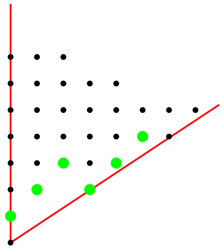

- Equivalent to the computation of several Hilbert bases computations.

Graver Bases Algorithms

- We are interested on optimization of a convex function over $\{x \in \mathbf{Z}^n : Ax = b,\ x \geq 0\}$. We will use basic Algebraic Geometry.
- For the lattice $L(A) = \{x \in \mathbf{Z}^n : Ax = 0\}$ introduce a natural partial order on the lattice vectors.
- For $u, v \in \mathbf{Z}^n$. u is *conformally smaller* than v, denoted $u \sqsubset v$, if $|u_i| \leq |v_i|$ and $u_i v_i \geq 0$ for $i = 1, \ldots, n$.
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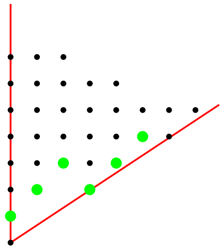

- Equivalent to the computation of several Hilbert bases computations.

- The Graver basis of an integer matrix A is the set of conformal-minimal nonzero integer dependencies on A.

- **Example:** If $A = [1\ 2\ 1]$ then its Graver basis is

$$\pm\{[2,-1,0],[0,-1,2],[1,0,-1],[1,-1,1]\}$$

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- The fastest algorithm to compute Graver bases is based on a completion and project-and-lift method (Got Groebner bases?). Implemented in `4ti2` (by R. Hemmecke and P. Malkin).

- Graver bases contain, and generalize, the LP test set given by the circuits of the matrix A. Circuits contain all possible edges of polyhedra in the family

$$P(b) := \{x \mid Ax = b, x \geq 0\}$$

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- **Theorem** The Graver basis contains all edges for all integer hulls $conv(\{x \mid Ax = b, x \geq 0,\ x \in \mathbf{Z}^n\})$ as b changes.

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- Here is how to construct it, consider

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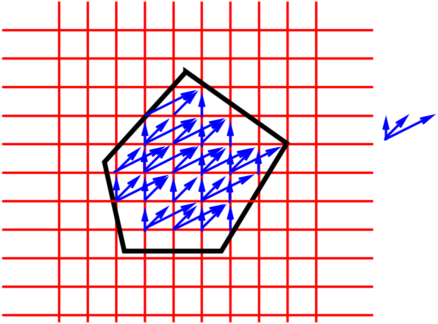
Nodes are lattice points in $L(b)$ and the Graver basis elements give directed edges departing from each lattice point $u \in L(b)$.



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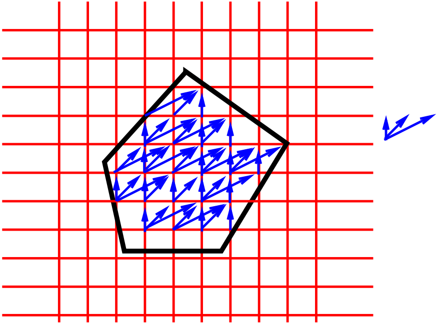

GOOD NEWS: Test Sets and Augmentation Method

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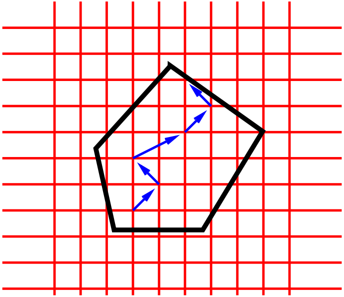

- **Theorem** [J. Graver 1975] Graver bases for A can be used to solve the augmentation problem Given $A \in \mathbf{Z}^{m \times n}$, $x \in \mathbf{N}^n$ and $c \in \mathbf{Z}^n$, either find an improving direction $g \in \mathbf{Z}^n$, namely one with $x - g \in \{y \in \mathbf{N}^n : Ay = Ax\}$ and $cg > 0$, or assert that no such g exists.

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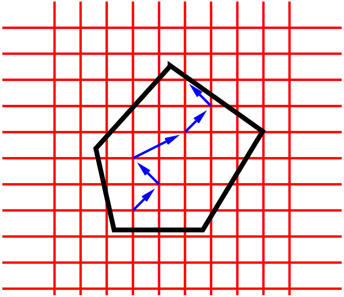

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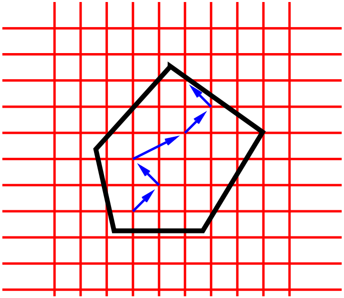

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BAD NEWS!!

- Graver bases contain a Gröbner bases, Hilbert bases. Work by Hosten, Graver, Scarf, Sturmfels, Sullivant, Thomas, Weismantel et al. and many others.
- Graver test sets can be exponentially large even in fixed dimension! Very hard to compute, you don't want to do this too often.
- People typically stored as a list of the whole test set and has to search within.
- NP-complete problem to decide whether a list of vectors is a complete Graver bases.
- New Results: There are useful cases where Graver bases become very manageable and efficient.
- BUT WE NEED HIGHLY STRUCTURED MATRICES!!

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Special Assumption II : Highly structured Matrices

Fix any pair of integer matrices A and B with the same number of columns, of dimensions $r \times q$ and $s \times q$, respectively. The n-fold matrix of the ordered pair A, B is the following $(s + nr) \times nq$ matrix,

$$[A, B]^{(n)} \quad := \quad (\mathbf{1}_n \otimes B) \oplus (I_n \otimes A) \quad = \quad \begin{pmatrix} B & B & B & \cdots & B \\ A & 0 & 0 & \cdots & 0 \\ 0 & A & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A \end{pmatrix} \quad .$$

N-fold systems DO appear in applications! Yes, Transportation problems with fixed number of suppliers!

Theorem Fix any integer matrices A, B of sizes $r \times q$ and $s \times q$, respectively. Then there is a polynomial time algorithm that, given any n, an integer vectors b, cost vector c, and a convex function f, solves the corresponding n-fold integer programming problem.

$$\max\{f(cx) : \ [A, B]^{(n)}x = b, \ x \in \mathbf{N}^{nq}\} \quad .$$

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- **Key Lemma** Fix any pair of integer matrices $A \in \mathbf{Z}^{r \times q}$ and $B \in \mathbf{Z}^{s \times q}$. Then there is a polynomial time algorithm that, given n, computes the Graver basis $G([A, B]^{(n)})$ of the n-fold matrix $[A, B]^{(n)}$. In particular, the cardinality and the bit size of $G([A, B]^{(n)})$ are bounded by a polynomial function of n.
- **Key Idea (from Algebraic Geometry)** [Aoki-Takemura, Santos-Sturmfels, Hosten-Sullivant] For every pair of integer matrices $A \in \mathbf{Z}^{r \times q}$ and $B \in \mathbf{Z}^{s \times q}$, there exists a constant $g(A, B)$ such that for all n, the Graver basis of $[A, B]^{(n)}$ consists of vectors with at most $g(A, B)$ the number nonzero components.

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Proof by Example

Consider the matrices $A = [1\ 1]$ and $B = I_2$. The Graver complexity of the pair A, B is $g(A, B) = 2$.

$$[A,B]^{(2)} \;=\; \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad G([A,B]^{(2)}) \;=\; \pm \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} \quad .$$

By our theorem, the Graver basis of the 4-fold matrix

$$[A,B]^{(4)} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

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Conclusions and Future work

- LINEAR Methods are not sufficient to solve all current integer optimization models, even the simple linear ones!
- There is demand to solve NON-LINEAR optimization problems, not just model things linearly anymore.
- In fact NON-LINEAR ideas can be applied in classical problems too! (ASK ME about them!):
    - Hilbert's Nullstellensatz Algorithm in Graph Optimization problems
    - Central Paths of Interior point methods as Algebraic Curves
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Merci

Thank you

Gracias