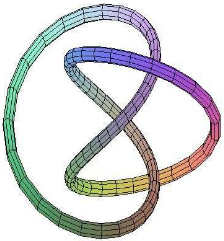
The Convex Hull of a Space Curve

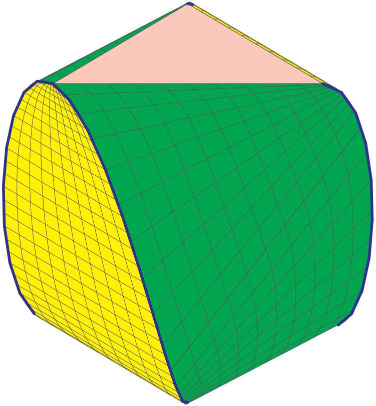
Bernd Sturmfels, UC Berkeley
(joint work with Kristian Ranestad)



Convex Algebraic Geometry Seminar
UC Berkeley, January 20, 2010

Convex Hull of a Trigonometric Curve

$$\{\, (\cos(\theta),\, \sin(2\theta),\, \cos(3\theta)\,) \in \mathbb{R}^3 \,:\, \theta \in [0, 2\pi] \,\}$$



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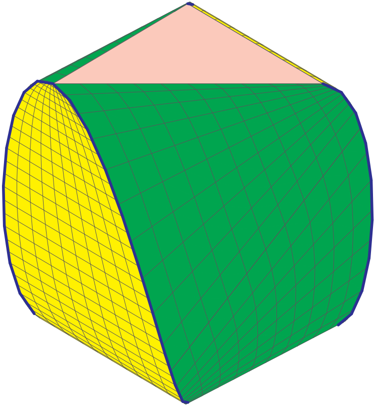
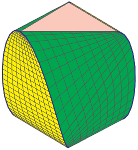


$$= \{\, (x, y, z) \in \mathbb{R}^3 \,:\, x^2 - y^2 - xz \;=\; z - 4x^3 + 3x \;=\; 0 \,\}$$

Lifted LMI Representation

Convex hulls of rational curves are projected spectrahedra, so they can be expressed in terms of Linear Matrix Inequalities.



$$= \left\{ (x,y,z) \in \mathbb{R}^3 \mid \exists\, u,v,w \in \mathbb{R} : \right.$$

$$\left. \begin{pmatrix} 1 & x+ui & v+yi & z+wi \\ x-ui & 1 & x+ui & v+yi \\ v-yi & x-vi & 1 & x+ui \\ z-wi & v-yi & x-ui & 1 \end{pmatrix} \succeq 0 \right\}.$$

Here $i = \sqrt{-1}$ and "$\succeq 0$" means that this Hermitian 4×4-matrix is positive semidefinite.

Boundary Surface Patches

The yellow surface has degree 3 and is defined by

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The green surface has degree **16** and its defining polynomial is

$$\begin{gathered}
1024x^{16} - 12032x^{14}y^2 + 52240x^{12}y^4 - 96960x^{10}y^6 + 56160x^8y^8 + 19008x^6y^{10} + 1296x^4y^{12} + 6144x^{15}z \\
-14080x^{13}y^2z - 72000x^{11}y^4z + 149440x^9y^6z + 79680x^7y^8z + 7488x^5y^{10}z + 15360x^{14}z^2 + 36352x^{12}y^2z^2 \\
+151392x^{10}y^4z^2 + 131264x^8y^6z^2 + 18016x^6y^8z^2 + 20480x^{13}z^3 + 73216x^{11}y^2z^3 + 105664x^9y^4z^3 + 23104x^7y^6z^3 \\
+15360x^{12}z^4 + 41216x^{10}y^2z^4 + 16656x^8y^4z^4 + 6144x^{11}z^5 + 6400x^9y^2z^5 + 1024x^{10}z^6 - 26048x^{14} - 135688x^{12}y^2 \\
+178752x^{10}y^4 + 124736x^8y^6 - 210368x^6y^8 + 792x^4y^{10} + 5184x^2y^{12} + 432y^{14} - 77888x^{13}z + 292400x^{11}y^2z \\
+10688x^9y^4z - 492608x^7y^6z - 67680x^5y^8z + 21456x^3y^{10}z + 2592xy^{12}z - 81600x^{12}z^2 - 65912x^{10}y^2z^2 \\
-464256x^8y^4z^2 - 192832x^6y^6z^2 + 31488x^4y^8z^2 + 6552x^2y^{10}z^2 - 40768x^{11}z^3 - 194400x^9y^2z^3 - 196224x^7y^4z^3 \\
+14912x^5y^6z^3 + 8992x^3y^8z^3 - 20800x^{10}z^4 - 84088x^8y^2z^4 - 7360x^6y^4z^4 + 7168x^4y^6z^4 - 12480x^9z^5 - 9680x^7y^2z^5 \\
+3264x^5y^4z^5 - 2624x^8z^6 + 760x^6y^2z^6 + 64x^7z^7 + 189649x^{12} + 104700x^{10}y^2 - 568266x^8y^4 + 268820x^6y^6 \\
+118497x^4y^8 - 42984x^2y^{10} - 432y^{12} + 62344x^{11}z - 592996x^9y^2z + 421980x^7y^4z + 377780x^5y^6z - 79748x^3y^8z \\
-18288xy^{10}z + 104620x^{10}z^2 + 56876x^8y^2z^2 + 480890x^6y^4z^2 - 12440x^4y^6z^2 - 51354x^2y^8z^2 - 936y^{10}z^2 \\
+35096x^9z^3 + 181132x^7y^2z^3 + 73800x^5y^4z^3 - 52792x^3y^6z^3 - 3780xy^8z^3 - 6730x^8z^4 + 52596x^6y^2z^4 \\
-19062x^4y^4z^4 - 5884x^2y^6z^4 + y^8z^4 + 6008x^7z^5 + 2516x^5y^2z^5 - 4324x^3y^4z^5 + 4xy^6z^5 + 2380x^6z^6 \\
-1436x^4y^2z^6 + 6x^2y^4z^6 - 152x^5z^7 + 4x^3y^2z^7 + x^4z^8 - 305250x^{10} + 313020x^8y^2 + 174078x^6y^4 \\
-291720x^4y^6 + 74880x^2y^8 + 84400x^9z + 278676x^7y^2z - 420468x^5y^4z + 20576x^3y^6z + 40704xy^8z \\
-25880x^8z^2 - 76516x^6y^2z^2 - 148254x^4y^4z^2 + 77840x^2y^6z^2 + 5248y^8z^2 - 29808x^7z^3 - 49388x^5y^2z^3 \\
+23080x^3y^4z^3 + 14560xy^6z^3 + 14420x^6z^4 - 7852x^4y^2z^4 + 9954x^2y^4z^4 + 568y^6z^4 + 848x^5z^5 + 92x^3y^2z^5 \\
+1164xy^4z^5 - 984x^4z^6 + 724x^2y^2z^6 - 2y^4z^6 + 112x^3z^7 - 4xy^2z^7 - 2x^2z^8 + 140625x^8 - 270000x^6y^2 \\
+172800x^4y^4 - 36864x^2y^6 - 75000x^7z + 36000x^5y^2z + 46080x^3y^4z - 24576xy^6z - 12500x^6z^2 \\
+49200x^4y^2z^2 - 19968x^2y^4z^2 - 4096y^6z^2 + 15000x^5z^3 - 10560x^3y^2z^3 - 3072xy^4z^3 \\
-2250x^4z^4 - 1872x^2y^2z^4 + 768y^4z^4 - 520x^3z^5 + 672xy^2z^5 + 204x^2z^6 - 48y^2z^6 - 24xz^7 + z^8.
\end{gathered}$$

Basic Definitions

Let C be a compact real algebraic curve in $\mathbb{R}^3$ and $\bar{C}$ its Zariski closure of C in $\mathbb{CP}^3$. We define the *degree* and *genus* of C by way of the complex projective curve $\bar{C} \subset \mathbb{CP}^3$:

$$d = \mathrm{degree}(C) := \mathrm{degree}(\bar{C}) \text{ and } g = \mathrm{genus}(C) := \mathrm{genus}(\bar{C}).$$

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The convex hull $\mathrm{conv}(C)$ of the real algebraic curve C is a compact, convex, semi-algebraic subset of $\mathbb{R}^3$, and its boundary $\partial\mathrm{conv}(C)$ is a pure 2-dimensional semi-algebraic subset of $\mathbb{R}^3$.

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Let K be the subfield of $\mathbb{R}$ over which the curve C is defined. The *algebraic boundary* of $\mathrm{conv}(C)$ is the K-Zariski closure of $\partial\mathrm{conv}(C)$ in $\mathbb{C}^3$. The algebraic boundary is denoted $\partial_a\mathrm{conv}(C)$. This complex surface is usually reducible and reduced. Its defining polynomial in $K[x, y, z]$ is unique up to scaling.

Degree Formula for Smooth Curves

Theorem. Let C be a general smooth compact curve of degree d and genus g in $\mathbb{R}^3$. The algebraic boundary $\partial_a \mathrm{conv}(C)$ of its convex hull is the union of the edge surface of degree $2(d-3)(d+g-1)$ and the tritangent planes of which there are $8\binom{d+g-1}{3} - 8(d+g-4)(d+2g-2) + 8g - 8$.

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A plane H in $\mathbb{CP}^3$ is a *tritangent plane* of $\bar{C}$ if H is tangent to $\bar{C}$ at three points. We count these using *De Jonquières' formula*.

Given points $p_1, p_2 \in C$, their secant line $L = \mathrm{span}(p_1, p_2)$ is a *stationary bisecant* if the tangent lines of C at p_1 and p_2 lie in a common plane. The *edge surface* of C is the union of all stationary bisecant lines. Its degree was determined by Arrondo *et al.* (2001).

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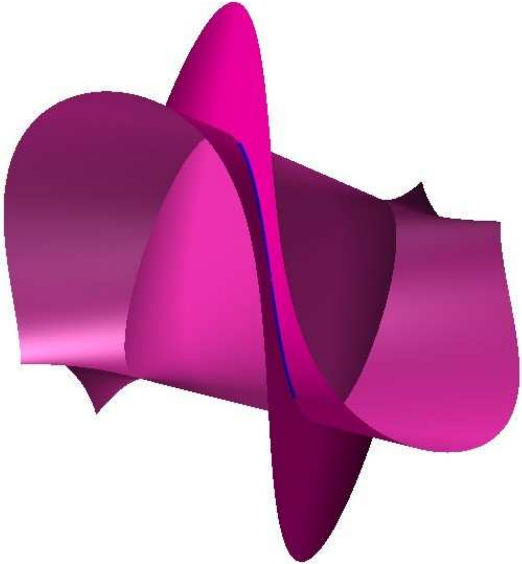
Example. If $d = 4$ and $g = 0$ then the two numbers are 6 and 0.

Edge Surface of a Rational Quartic Curve



... is irreducible of degree six. $d = 4, g = 0$

Edge Surface of an Elliptic Curve

The intersection $C = Q_1 \cap Q_2$ of two general quadratic surfaces is an *elliptic curve*: it has genus $g = 1$ and degree $d = 4$. The edge surface of C has degree 8. It is the union of four quadratic cones.

Proof: The pencil of quadrics $Q_1 + tQ_2$ contains four singular quadrics, corresponding to the four real roots t_1, t_2, t_3, t_4 of $f(t) = \det(Q_1 + tQ_2)$. The stationary bisecants to C are the rulings of these cones. The defining polynomial of $\partial_a \mathrm{conv}(C)$ is

$$\prod_{i=1}^{4}(Q_1 + t_i Q_2)(x,y,z) \;=\; \mathrm{resultant}_t\big(\, f(t),\, (Q_1 + tQ_2)(x,y,z)\,\big).$$

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Conclusion: The edge surface of a curve $C \subset \mathbb{R}^3$ can have multiple components even if $\bar{C} \subset \mathbb{CP}^3$ is smooth and irreducible.

Conjecture: At most one of these components is not a cone.

Trigonometric Curves

A *trigonometric polynomial* of degree d is an expression of the form

$$f(\theta) \;=\; \sum_{j=1}^{d/2} \alpha_j \cos(j\theta) \;+\; \sum_{j=1}^{d/2} \beta_j \sin(j\theta) \;+\; \gamma.$$

Here d is even. A *trigonometric space curve* of degree d is a curve parametrized by three such trigonometric polynomials:

$$C \;=\; \bigl\{ \bigl(f_1(\theta), f_2(\theta), f_3(\theta)\bigr) \in \mathbb{R}^3 \,:\, \theta \in [0, 2\pi] \bigr\}.$$

For general $\alpha_j, \beta_j, \gamma \in \mathbb{R}$, the curve $\bar{C} \subset \mathbb{CP}^3$ is smooth and $g = 0$.

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For general $\alpha_j, \beta_j, \gamma \in \mathbb{R}$, the curve $\bar{C} \subset \mathbb{CP}^3$ is smooth and $g = 0$. We get a parametrization $\mathbb{CP}^1 \to \bar{C}$ by the change of coordinates

$$\cos(\theta) = \frac{x_0^2 - x_1^2}{x_0^2 + x_1^2} \quad \text{and} \quad \sin(\theta) = \frac{2x_0x_1}{x_0^2 + x_1^2}.$$

Substituting into the right hand side of the equation

$$\begin{pmatrix} \cos(j\theta) & \sin(j\theta) \\ -\sin(j\theta) & \cos(j\theta) \end{pmatrix} \;=\; \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}^{j},$$

this change of variables expresses $\cos(j\theta)$ and $\sin(j\theta)$ as homogeneous rational functions of degree 0 in $(x_0 : x_1)$.

Rational Sextic Curves

Fix $d = 6, g = 0$. The algebraic boundary of the convex hull of a *general* trigonometric curve of degree 6 consists of 8 tritangent planes and an irreducible edge surface of degree 30. For special curves, these degrees drop and the edge surface degenerates...

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Barth and Moore (1988) constructed the following sextic curve $\bar{C}$:

$$\mathbb{CP}^1 \to \mathbb{CP}^3, \ (x_0 : x_1) \mapsto (\, x_0^6 - 2x_0x_1^5 \ : \ 2x_0^5x_1 + x_1^6 \ : \ x_0^4x_1^2 \ : \ x_0^2x_1^4 \,).$$

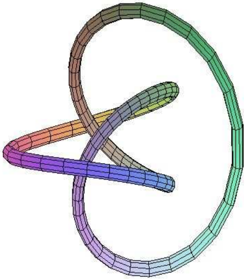
This curve is smooth, so its edge surface should have the expected degree 30. However, when we run our algorithm, the output is

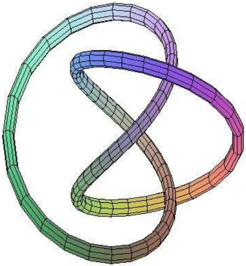
$$27x^5y^5 + 3125y^{10} - 1875x^2y^7z + \cdots + 27z^5 - 16y^3z.$$

The edge surface of $\bar{C}$ carries a non-reduced structure of multiplicity three. All tritangent planes of $\bar{C}$ have collinear tangency points and all stationary bisecants are tritangents.

Morton's Curve

$$C \quad : \quad \theta \quad \mapsto \quad \frac{1}{2 - \sin(2\theta)} \left(\cos(3\theta), \sin(3\theta), \cos(2\theta) \right)$$





Freedman (1980) whether every knotted curve in $\mathbb{R}^3$ must have a tritangent plane. Morton (1991) showed that the answer is NO.

Tritangent Planes via Chow Forms

Let $\bar{C}$ be a rational projective curve of degree d defined by

$$\mathbb{CP}^1 \to \mathbb{CP}^3\,,\ x = (x_0 : x_1) \mapsto (F_0(x) : F_1(x) : F_2(x) : F_3(x))$$

A plane $\{\alpha + \beta x + \gamma y + \delta z = 0\}$ in $\mathbb{CP}^3$ correspond to a point $(\alpha : \beta : \gamma : \delta)$ in the dual projective space $(\mathbb{CP}^3)^*$. The plane is tangent to $\bar{C}$ at a point p if its preimage $(x_{p,0} : x_{p,1}) \in \mathbb{CP}^1$ is a double root of the binary form

$$\alpha F_0(x) + \beta F_1(x) + \gamma F_2(x) + \delta F_3(x). \tag{1}$$

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$$\alpha F_0(x) \, + \, \beta F_1(x) \, + \, \gamma F_2(x) \, + \delta F_3(x). \tag{1}$$

Our algorithm computes

$$\mathcal{T}_C \;=\; \big\{ (\alpha : \beta : \gamma : \delta) \in (\mathbb{CP}^3)^* \;:\; \text{(1) has three double roots} \big\}.$$

This set has cardinality $8\binom{d-3}{d}$ and is represented by its *Chow form*

$$\prod_{(\alpha:\beta:\gamma:\delta)\in\mathcal{T}_C} (\alpha + \beta x + \gamma y + \delta z).$$

If the $F_i(x)$ have coefficients in $\mathbb{Q}$ then so does the Chow form. The corresponding surface is the union of all tritangent planes.

Morton's Curve

Morton's curve has the polynomial parametrization $\mathbb{CP}^1 \to \mathbb{CP}^3$,

$$(x_0 : x_1) \;\mapsto\; \begin{pmatrix} 2(x_0^4 + 2x_0^2x_1^2 + x_1^4 - 2x_0^3x_1 + 2x_0x_1^3)(x_0^2 + x_1^2) \\ (x_0 - x_1)(x_0 + x_1)(x_1^2 + 4x_0x_1 + x_0^2)(x_1^2 - 4x_0x_1 + x_0^2) \\ 2x_0x_1(x_0^2 - 3x_1^2)(3x_0^2 - x_1^2) \\ (2x_0x_1 + x_0^2 - x_1^2)(x_0^2 - x_1^2 - 2x_0x_1)(x_0^2 + x_1^2) \end{pmatrix}$$

The Chow form of the tritangent planes equals $(x^2 + y^2)^2$ times

$$\begin{array}{c} 13225x^4 + 58880x^3y + 91986x^2y^2 - 638976x^2z^2 + 13225y^4 \\ +58880xy^3 - 1148160xyz^2 - 638976y^2z^2 + 6230016z^4 + 449280x^2z \\ -449280y^2z - 409600x^2 - 736000xy - 409600y^2 - 7987200z^2 + 2560000 \end{array}$$

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This quartic is irreducible over $\mathbb{Q}$ but factors over $\mathbb{R}$:

$$(1 + 0.3393\,x + 0.2118\,y + 1.2489\,z) \cdot (1 - 0.3393\,x - 0.2118\,y + 1.2489\,z) \cdot$$
$$(1 + 0.2118\,x + 0.3393\,y - 1.2489\,z) \cdot (1 - 0.2118\,x - 0.3393\,y - 1.2489\,z).$$

Each of these four planes touches the curve $\bar{C}$ in one real point and two imaginary points. This answers Freedman's question. Hence $\partial_a \mathrm{conv}(C)$ consists only of the edge surface of C. It decomposes (over $\mathbb{Q}$) into two components of degrees 10 and 20.

Curves with Singularities

Theorem. The edge surface of a general irreducible space curve of degree d, geometric genus g, with n ordinary nodes and k ordinary cusps, has degree $2(d-3)(d+g-1) - 2n - 2k$. The cone of bisecants through each cusp has degree $d-2$ and is a component of the edge surface.

Here the singularity is called *ordinary* if no plane in $\mathbb{CP}^3$ intersects the curve with multiplicity more than 4.

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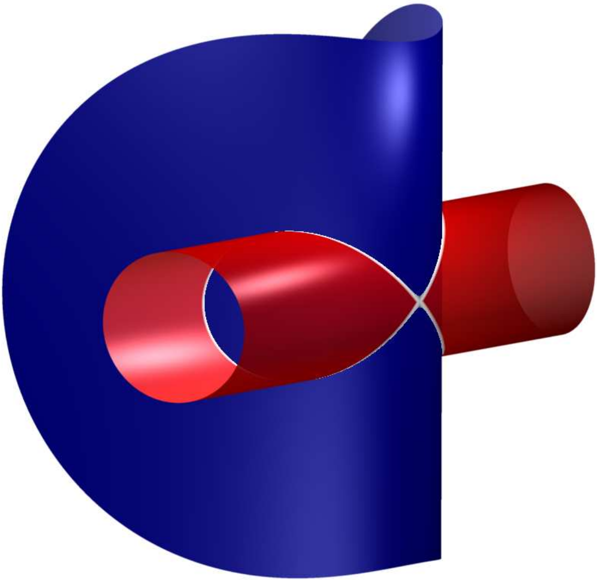
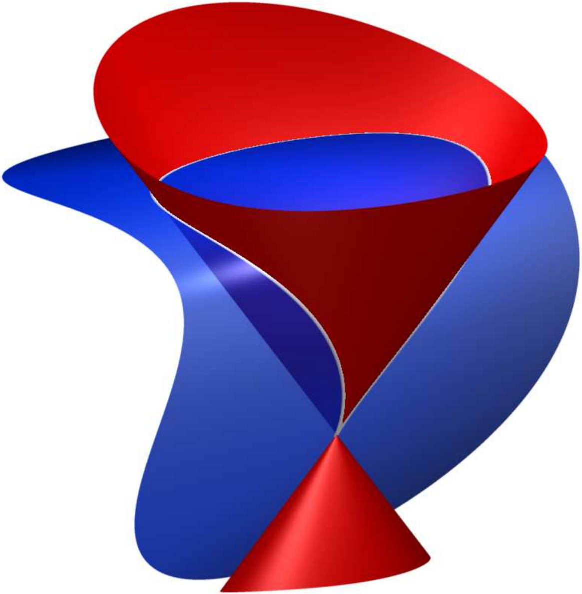
Example. $(d = 4, g = 0, n + k = 1)$
Consider a rational quartic curve with one ordinary singular point. The edge surface has degree 4. It is the union of two quadric cones whose intersection equals the curve. If the singularity is an ordinary cusp then one of the two quadrics has its vertex at the cusp.

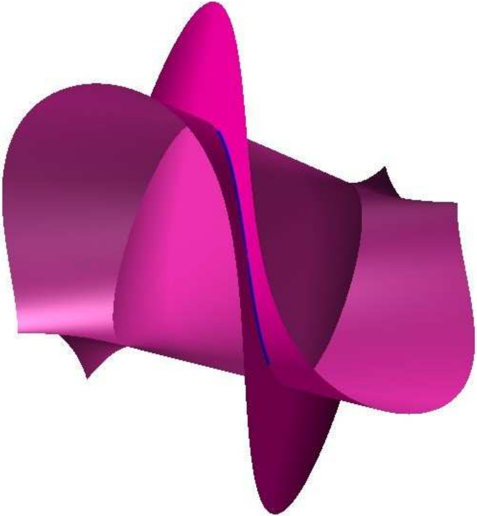
Rational Quartic with an Ordinary Node



Rational Quartic with an Ordinary Cusp



Smooth Rational Quartic Curve



The edge surface of the curve $(\cos(\theta), \sin(\theta) + \cos(2\theta), \sin(2\theta))$ is irreducible of degree six.

$d = 4, g = 0, n + k = 0$

Take-Home Messages

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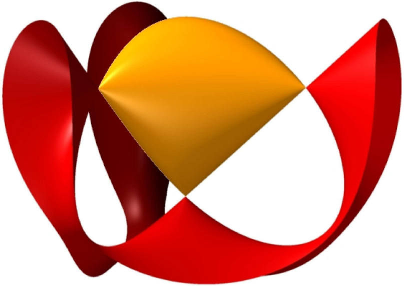

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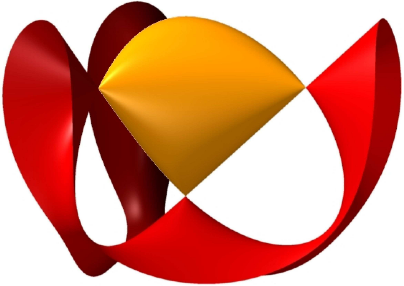

- There are 3-dimensional convex bodies other than this pillow.
- I will teach *Topics in Applied Math* (Math 275) in Fall '10.