Quartic Symmetroids and Spectrahedra

Cynthia Vinzant,
University of Michigan

with
John Christian Ottem,
Kristian Ranestad, and
Bernd Sturmfels
A quartic symmetroid is a surface $\mathcal{V}(f) \subset \mathbb{P}^3(\mathbb{C})$ given by

$$f = \det(A(x)) = \det(x_0A_0 + x_1A_1 + x_2A_2 + x_3A_3)$$

where $A_0, A_1, A_2, A_3$ are $4 \times 4$ symmetric matrices.

Fun facts:

- $\mathcal{V}(f)$ has 10 nodes (of rank 2)
- co-dimension 10 in $\mathbb{P}(\mathbb{C}[x_0, x_1, x_2, x_3]_4)$
- studied by Cayley in a set of memoirs 1869 - 1871
Here I’ll talk about in surfaces $\mathcal{V}(\det(A(x)))$ where

- the matrices $A_0, A_1, A_2, A_3$ are real and
- their span contains a positive definite matrix.

**Motivation 1:** The convex sets $\{x \in \mathbb{R}^4 : A(x) \succeq 0\}$ appear as feasible sets (spectrahedra) in semidefinite programming.

**Motivation 2:** Having a positive definite matrix puts interesting constraints on the surface $\mathcal{V}_\mathbb{R}(f)$.

For example . . .

Friedland *et. al.* (1984) showed that in this case $\mathcal{V}(\det(A(x)))$ has a real node.
Let $A_0, A_1, \ldots, A_n$ be real symmetric $d \times d$ matrices and

$$A(x) = x_0 A_0 + x_1 A_1 + \ldots + x_n A_n.$$ 

**Spectrahedron:**

$$\{ x \in \mathbb{R}^{n+1} : A(x) \text{ is positive semidefinite} \}$$

**projectivize**

$$\{ x \in \mathbb{P}^n(\mathbb{R}) : A(x) \text{ is semidefinite} \}$$

(bounded by the hypersurface $\mathcal{V}(\det(A(x)))$)

**Example:**

$$A(x) = \begin{pmatrix}
    x_0 + x_1 & x_2 \\
    x_1 & x_0 - x_1
\end{pmatrix}$$

**Goal:** Understand the algebraic and topological properties of spectrahedra and their bounding polynomials.
Polynomials bounding spectrahedra

Spectrahedra are bounded by hyperbolic polynomials, $\det(A(x))$.

A polynomial $f$ is hyperbolic with respect to a point $p$ if every real line through $p$ meets $\mathcal{V}(f)$ in only real points.

Theorem (Helton-Vinnikov 2007). A polynomial $f \in \mathbb{R}[x_0, x_1, x_2]_d$ bounds a spectrahedron if and only if $f$ is hyperbolic.
Spectrahedra and interlacers

The diagonal \((d - 1) \times (d - 1)\) minors of \(A(x)\) interlace the determinant \(\det(A(x))\).

**Theorem (Plaumann-V. 2013).** The matrix \(A(x)\) is definite at some point if and only if its minors interlace the determinant.
The variety of rank-\((d - 2)\) matrices in \(\mathbb{C}^{d \times d}_{\text{sym}}\) has codimension \(3\) and degree \(\binom{d+1}{3}\).

Generically, the \(\text{span}_{\mathbb{C}}\{A_0, A_1, A_2, A_3\}\) meets this variety \textit{transversely} and contains \(\binom{d+1}{3}\) matrices of rank \(d - 2\).

The complex surface \(\mathcal{V}(\det(A(x)))\) bounding a three-dimensional spectrahedron has \(\binom{d+1}{3}\) nodes.
Over $\mathbb{C}$ there are (generically) 4 nodes of rank one.

Either 2 or 4 of them are real and lie on the spectrahedron.
Over $\mathbb{C}$ there are generically 10 nodes of rank two.

There are two flavors of real node (on or off the spectrahedron). What configurations are possible?
Theorem (Degtyarev-Itenberg, 2011)

There is a (transversal) quartic spectrahedron with $\alpha$ nodes on its boundary and $\beta$ nodes on its real surface if and only if

$\alpha, \beta$ are even and $2 \leq \alpha + \beta \leq 10$.

\[\begin{align*}
\alpha &= 8 \\
\beta &= 2 \\
\alpha &= 0 \\
\beta &= 10 \\
\alpha &= 2 \\
\beta &= 0
\end{align*}\]
Idea of Cayley: Look at the projection of $\mathcal{V}(f)$ from a node $p$.

This projection $\pi_p : \mathcal{V}(f) \to \mathbb{P}^2$ from a node $p$ is a double cover of $\mathbb{P}^2$ whose branch locus is a sextic curve.

Why? If $p = [1:0:0:0]$ then

$$f = a \cdot x_0^2 + b \cdot x_0 + c \quad \text{where} \quad a, b, c \in \mathbb{R}[x_1, x_2, x_3].$$

The branch locus of $\pi_p$ is $\mathcal{V}(b^2 - 4ac)$. 
Theorem (Cayley 1869-71)

A quartic $f \in \mathbb{C}[x_0, x_1, x_2, x_3]_4$ with node $p$ is a symmetroid if and only if the branch locus of $\pi_p$ is the product of two cubics, $b_1 \cdot b_2$.

Moreover the images of the other 9 nodes are $\mathcal{V}(b_1) \cap \mathcal{V}(b_2)$. 
For \( p \in \text{Spec} \), \( b_1 = \overline{b_2} \).
The image \( \pi_p(\text{Spec}) \) is the conic \( \{a \geq 0\} \).

For \( p \notin \text{Spec} \), \( b_1, b_2 \) are real and hyperbolic.
The image \( \pi_p(\text{Spec}) \) is the intersection of cubic ovals.
If \( p = [1 : 0 : 0 : 0] \) and \( A(x) = x_0 \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \), then the branch cubics \( b_1, b_2 \) are diagonal minors of \( A(0, x_1, x_2, x_3) \).
The view from a node: interlacing branch locus

The image of the spectrahedron is the intersection of cubic ovals.

There are an even number of spectrahedral nodes.

To understand the other direction of the Degtyarev-Itenberg Theorem ...
\((A_0 \ A_1 \ A_2 \ A_3)\) giving different types of spectrahedra

| (2, 2) | \( \begin{bmatrix} 3 & 4 & 1 & -4 \\ 4 & 14 & -6 & -10 \\ 1 & -6 & 9 & 2 \\ -4 & -10 & 2 & 8 \end{bmatrix} \) | \( \begin{bmatrix} 11 & 0 & 2 & 2 \\ 0 & 6 & -1 & 4 \\ 2 & -1 & 6 & 2 \\ 2 & 4 & 2 & 4 \end{bmatrix} \) | \( \begin{bmatrix} 17 & -3 & 2 & 9 \\ -3 & 6 & -4 & 1 \\ 2 & -4 & 13 & 10 \\ 9 & 1 & 10 & 17 \end{bmatrix} \) | \( \begin{bmatrix} 9 & -3 & 9 & 3 \\ -3 & 10 & 6 & -7 \\ 9 & 6 & 18 & -3 \\ 3 & -7 & -3 & 5 \end{bmatrix} \) |
| (4, 4) | \( \begin{bmatrix} 18 & 3 & 9 & 6 \\ 3 & 5 & -1 & -3 \\ 9 & -1 & 13 & 7 \\ 6 & -3 & 7 & 6 \end{bmatrix} \) | \( \begin{bmatrix} 17 & -10 & 4 & 3 \\ -10 & 14 & -1 & -3 \\ 4 & -1 & 5 & -4 \\ 3 & -3 & -4 & 6 \end{bmatrix} \) | \( \begin{bmatrix} 8 & 6 & 10 & 10 \\ 6 & 18 & 6 & 15 \\ 10 & 6 & 14 & 9 \\ 10 & 15 & 9 & 22 \end{bmatrix} \) | \( \begin{bmatrix} 8 & -4 & 8 & 0 \\ -4 & 10 & -4 & 0 \\ 8 & -4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \) |
| (6, 6) | \( \begin{bmatrix} 10 & 8 & 2 & 6 \\ 8 & 14 & 0 & 2 \\ 2 & 0 & 5 & 7 \\ 6 & 2 & 7 & 11 \end{bmatrix} \) | \( \begin{bmatrix} 11 & -6 & 10 & 9 \\ -6 & 10 & -5 & -5 \\ 10 & -5 & 14 & 11 \\ 9 & -5 & 11 & 9 \end{bmatrix} \) | \( \begin{bmatrix} 6 & 2 & 6 & -5 \\ 6 & 2 & 6 & -5 \\ 6 & 2 & 6 & -5 \\ -5 & 0 & -5 & 5 \end{bmatrix} \) | \( \begin{bmatrix} 8 & 6 & 2 & -2 \\ 6 & 9 & 9 & 6 \\ 2 & 9 & 13 & 12 \\ -2 & 6 & 12 & 13 \end{bmatrix} \) |
| (8, 8) | \( \begin{bmatrix} 5 & 3 & -3 & -4 \\ 3 & 6 & -3 & -2 \\ 3 & -3 & 6 & 4 \\ -4 & -2 & 4 & 4 \end{bmatrix} \) | \( \begin{bmatrix} 19 & 10 & 12 & 17 \\ 10 & 14 & 10 & 7 \\ 12 & 10 & 10 & 11 \\ 17 & 7 & 11 & 17 \end{bmatrix} \) | \( \begin{bmatrix} 5 & 1 & 3 & -3 \\ 1 & 5 & -7 & 1 \\ 3 & -7 & 22 & 7 \\ -3 & -1 & 7 & 10 \end{bmatrix} \) | \( \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 4 & 4 \\ 2 & 2 & 4 & 8 \end{bmatrix} \) |
| (10, 10) | \( \begin{bmatrix} 18 & 6 & 6 & -6 \\ 6 & 2 & 2 & -2 \\ 6 & 2 & 2 & -2 \\ -6 & -2 & -2 & 4 \end{bmatrix} \) | \( \begin{bmatrix} 4 & -6 & 6 & 4 \\ -6 & 13 & -9 & -8 \\ 6 & -9 & 9 & 6 \\ 4 & -8 & 6 & 5 \end{bmatrix} \) | \( \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 4 & 0 & 6 \\ -3 & 0 & 9 & 0 \\ 0 & 6 & 0 & 9 \end{bmatrix} \) | \( \begin{bmatrix} 9 & -3 & 0 & 0 \\ -3 & 10 & 9 & 6 \\ 9 & 9 & 9 & -6 \\ 0 & -6 & -6 & 4 \end{bmatrix} \) |
| (2, 0) | \( \begin{bmatrix} 20 & 6 & -14 & -4 \\ 6 & 18 & 3 & -12 \\ -14 & 3 & 17 & -2 \\ -4 & -12 & -2 & 8 \end{bmatrix} \) | \( \begin{bmatrix} 54 & -27 & 16 & 12 \\ -27 & 18 & -2 & -15 \\ 16 & -2 & 20 & -10 \\ 12 & -15 & -10 & 21 \end{bmatrix} \) | \( \begin{bmatrix} 42 & -8 & 9 & -3 \\ -8 & 10 & 5 & -11 \\ 9 & 5 & 29 & 7 \\ -3 & -11 & 7 & 29 \end{bmatrix} \) | \( \begin{bmatrix} 0 & 9 & 3 & -3 \\ 9 & -9 & -6 & 6 \\ 3 & -6 & -3 & 3 \\ -3 & 6 & 3 & -3 \end{bmatrix} \) |
| (4, 2) | \( \begin{bmatrix} 9 & -4 & 1 & 1 \\ -4 & 5 & -3 & -2 \\ 1 & -3 & 3 & 1 \\ 1 & -2 & 1 & 1 \end{bmatrix} \) | \( \begin{bmatrix} 6 & 1 & 3 & 4 \\ 1 & 5 & 5 & 2 \\ 3 & 5 & 6 & 2 \\ 4 & 2 & 2 & 8 \end{bmatrix} \) | \( \begin{bmatrix} 8 & 2 & -6 & 4 \\ 2 & 5 & 1 & 3 \\ -6 & 1 & 6 & -2 \\ 4 & 3 & -2 & 3 \end{bmatrix} \) | \( \begin{bmatrix} -4 & 4 & -2 & 2 \\ 4 & 0 & 0 & -2 \\ -2 & 0 & 0 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix} \) |
| (6, 4) | \( \begin{bmatrix} 6 & -1 & 5 & 5 \\ 2 & 1 & 5 & -3 \\ 1 & 6 & 2 & 9 \\ 5 & -3 & 2 & 9 \end{bmatrix} \) | \( \begin{bmatrix} -5 & -5 & 5 & -3 \\ -5 & 6 & -5 & 5 \\ -3 & 5 & -3 & 9 \\ -3 & 5 & 3 & 2 \end{bmatrix} \) | \( \begin{bmatrix} 6 & -3 & 5 & 2 \\ -3 & 5 & -3 & 2 \\ 5 & -3 & 9 & -4 \\ 2 & 2 & -4 & 9 \end{bmatrix} \) | \( \begin{bmatrix} 0 & -2 & -2 & 0 \\ -2 & 1 & 2 & 1 \\ -2 & 2 & 3 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \) |
| (8, 6) | \( \begin{bmatrix} 4 & 0 & 4 & -2 \\ 0 & 5 & -2 & 5 \\ 4 & -2 & 8 & -4 \\ -2 & 5 & -4 & 6 \end{bmatrix} \) | \( \begin{bmatrix} 2 & 3 & -1 & -1 \\ 3 & 6 & -1 & -4 \\ -1 & -1 & 6 & -3 \\ -1 & -4 & -3 & 6 \end{bmatrix} \) | \( \begin{bmatrix} 6 & 2 & 0 & 1 \\ 2 & 8 & 0 & 1 \\ 0 & 4 & 8 & -2 \\ 1 & -2 & -2 & 1 \end{bmatrix} \) | \( \begin{bmatrix} -2 & 3 & 0 & 1 \\ -3 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 5 \end{bmatrix} \) |
| (10, 8) | \( \begin{bmatrix} 5 & -1 & -1 & 4 \\ -1 & 6 & -3 & 5 \\ -1 & -3 & 2 & -4 \\ 4 & 5 & -4 & 9 \end{bmatrix} \) | \( \begin{bmatrix} 8 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ -4 & -1 & 0 & 3 \end{bmatrix} \) | \( \begin{bmatrix} 6 & 5 & 1 & -2 \\ 5 & 9 & -3 & -4 \\ 1 & -3 & 6 & 4 \\ -2 & -4 & 4 & 4 \end{bmatrix} \) | \( \begin{bmatrix} 8 & 0 & 0 & -4 \\ 0 & 8 & 4 & 4 \\ 0 & 4 & 2 & 2 \\ -4 & 4 & 2 & 4 \end{bmatrix} \) |
### Combinatorial types of quartic spectrahedra (11-20)

| (4, 0) : | \begin{pmatrix} 21 & 10 & 1 & -6 \\ 10 & 10 & 0 & -1 \\ 1 & 0 & 2 & -3 \\ -6 & -1 & -3 & 6 \end{pmatrix} & | \begin{pmatrix} 0 & 6 & -6 & 2 \\ 6 & 3 & 0 & -4 \\ -6 & 0 & -3 & 5 \\ 2 & -4 & 5 & -3 \end{pmatrix} & | \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 2 & -1 & -1 & 5 \end{pmatrix} & | \begin{pmatrix} 0 & 3 & -1 & 1 \\ 3 & -3 & 8 & -5 \\ -1 & 8 & -5 & 4 \\ 1 & -5 & 4 & -3 \end{pmatrix} |
| (6, 2) : | \begin{pmatrix} 7 & -1 & 5 & 2 \\ -1 & 1 & -1 & 5 \\ 5 & -1 & 4 & 1 \\ 2 & 5 & 1 & 7 \end{pmatrix} & | \begin{pmatrix} -1 & -2 & 1 & -2 \\ -2 & -3 & 2 & -6 \\ 1 & 2 & 1 & 2 \\ -2 & -6 & 2 & -1 \end{pmatrix} & | \begin{pmatrix} 4 & 4 & 2 & -2 \\ 4 & 0 & 4 & -2 \\ 2 & 4 & 0 & -1 \\ -2 & -2 & -1 & 1 \end{pmatrix} & | \begin{pmatrix} -1 & 1 & 2 & 1 \\ -1 & -1 & -2 & -1 \\ 2 & -1 & -3 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix} |
| (8, 4) : | \begin{pmatrix} 16 & -4 & -16 & 10 \\ -4 & 18 & 0 & -13 \\ -16 & 0 & 20 & -9 \\ 10 & -13 & -9 & 19 \end{pmatrix} & | \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & -5 & 6 & 1 \\ -1 & 6 & -7 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} & | \begin{pmatrix} 0 & -16 & 0 & -8 \\ -16 & 0 & 16 & -16 \\ 0 & 16 & 0 & 8 \\ -8 & -16 & 8 & -16 \end{pmatrix} & | \begin{pmatrix} 16 & 16 & 3 \\ 9 & -9 & -12 & 9 \\ 16 & -12 & -15 & 15 \\ 3 & 9 & 15 & 0 \end{pmatrix} |
| (10, 6) : | \begin{pmatrix} 18 & -13 & 15 & 1 \\ -13 & 22 & 2 & -16 \\ 15 & 2 & 30 & -20 \\ 1 & -16 & -20 & 30 \end{pmatrix} & | \begin{pmatrix} -15 & 7 & 8 & 5 \\ 7 & -3 & -4 & -3 \\ 8 & -4 & -4 & -2 \\ -3 & -3 & -2 & 0 \end{pmatrix} & | \begin{pmatrix} 1 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -8 & -15 \\ -3 & 0 & -15 & -7 \end{pmatrix} & | \begin{pmatrix} -15 & 0 & -6 & 2 \\ 0 & 15 & 6 & 8 \\ -6 & 6 & 0 & 4 \\ 2 & 8 & 4 & 4 \end{pmatrix} |
| (6, 0) : | \begin{pmatrix} 3 & 6 & -4 & -4 \\ 6 & 13 & -5 & -5 \\ -4 & -5 & 19 & 20 \\ -4 & -5 & 20 & 23 \end{pmatrix} & | \begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 3 & 6 & 0 \\ -3 & 6 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & | \begin{pmatrix} 8 & 2 & -2 & 2 \\ 2 & -4 & -2 & 2 \\ -2 & -2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix} & | \begin{pmatrix} 1 & -2 & 1 & 3 \\ -2 & -5 & -11 & -15 \\ 1 & -11 & -8 & -6 \\ 3 & -15 & -6 & 0 \end{pmatrix} |
| (8, 2) : | \begin{pmatrix} 3 & -3 & 3 & -1 \\ -3 & 4 & -3 & 2 \\ 3 & -3 & 5 & 0 \\ -2 & 2 & 0 & 2 \end{pmatrix} & | \begin{pmatrix} -1 & 1 & -1 & -2 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix} & | \begin{pmatrix} 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & -4 \end{pmatrix} & | \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 3 & -1 & 2 \\ 1 & -1 & -1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} |
| (10, 4) : | \begin{pmatrix} 5 & -1 & -3 & 1 \\ -1 & 2 & 2 & 0 \\ -3 & 2 & 4 & -1 \\ 1 & 0 & -1 & 3 \end{pmatrix} & | \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & -4 & -2 \\ 0 & -4 & -4 & -2 \\ 0 & -2 & -2 & 0 \end{pmatrix} & | \begin{pmatrix} 0 & 4 & -4 & -6 \\ 4 & 0 & 2 & 1 \\ -4 & 2 & -4 & -4 \\ -6 & 1 & -4 & -3 \end{pmatrix} & | \begin{pmatrix} -3 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ -2 & 0 & -1 & -1 \end{pmatrix} |
| (8, 0) : | \begin{pmatrix} 9 & 0 & -7 & -10 \\ 0 & 5 & 0 & 2 \\ -7 & 0 & 15 & 5 \\ -10 & 2 & 5 & 13 \end{pmatrix} & | \begin{pmatrix} 8 & 6 & 5 & 8 \\ 6 & -8 & -5 & -4 \\ 5 & -5 & -3 & -2 \\ 8 & -4 & -2 & 0 \end{pmatrix} & | \begin{pmatrix} 8 & 4 & 11 & 4 \\ 4 & 0 & 10 & 0 \\ 11 & 10 & 5 & 10 \\ 4 & 0 & 10 & 0 \end{pmatrix} & | \begin{pmatrix} -4 & -4 & 2 & 4 \\ -4 & -4 & 2 & 4 \\ 2 & 2 & 0 & 0 \\ 4 & 4 & 0 & 0 \end{pmatrix} |
| (10, 2) : | \begin{pmatrix} 29 & -22 & 4 & -4 \\ -22 & 26 & -7 & 5 \\ 4 & -7 & 25 & -6 \\ -4 & 5 & -6 & 5 \end{pmatrix} & | \begin{pmatrix} -1 & -4 & -1 & -4 \\ 0 & -4 & -5 & -4 \\ -1 & -4 & -1 & -4 \\ -4 & -14 & 4 & -15 \end{pmatrix} & | \begin{pmatrix} -5 & 9 & 6 & 7 \\ 9 & 8 & -2 & 5 \\ 6 & -2 & -4 & -2 \\ 7 & 5 & -2 & 3 \end{pmatrix} & | \begin{pmatrix} -5 & 16 & -1 & -10 \\ 16 & -12 & 20 & 4 \\ -1 & 20 & 7 & 14 \\ -10 & 4 & -14 & 0 \end{pmatrix} |
| (10, 0) : | \begin{pmatrix} 51 & -34 & 5 & 60 \\ -34 & 147 & 30 & -37 \\ 5 & 30 & 99 & 40 \\ 60 & -37 & 40 & 135 \end{pmatrix} & | \begin{pmatrix} 15 & 97 & 64 & 36 \\ 97 & -13 & -50 & 76 \\ 64 & -50 & -63 & 40 \\ 36 & 76 & 40 & 48 \end{pmatrix} & | \begin{pmatrix} -27 & 45 & -27 & 51 \\ 95 & 0 & -30 & 10 \\ -27 & -30 & 48 & -44 \\ 51 & 10 & -44 & 24 \end{pmatrix} & | \begin{pmatrix} -60 & 30 & 10 & -52 \\ 30 & 45 & -55 & -2 \\ 10 & -55 & 40 & 32 \\ -52 & -2 & 32 & -32 \end{pmatrix} |
Many flavors of quartic spectrahedra
Non-generically, the span of $A_0, A_1, A_2, A_3$ might contain a curve of rank-two matrices.

\[
\begin{pmatrix}
  x_0 & 0 & 0 & 0 \\
  0 & x_1 & 0 & 0 \\
  0 & 0 & x_2 & 0 \\
  0 & 0 & 0 & x_3
\end{pmatrix}
\begin{pmatrix}
  x_0 + x_1 & x_2 & 0 & 0 \\
  x_2 & x_0 - x_1 & 0 & 0 \\
  0 & 0 & x_0 + x_3 & 0 \\
  0 & 0 & 0 & x_0 - x_3
\end{pmatrix}
\begin{pmatrix}
  x_0 & x_1 & x_2 & x_3 \\
  x_1 & x_0 & x_1 & x_2 \\
  x_2 & x_1 & x_0 & x_1 \\
  x_3 & x_2 & x_1 & x_0
\end{pmatrix}
\]
Spectrahedra can be understood using beautiful and classical algebraic geometry.

There is still lots to understand. What are the combinatorial types of spectrahedra of higher dimensions and degrees?

Thanks for your attention!