Numerical methods for computing real and complex tropical curves

Cynthia Vinzant
North Carolina State University

joint with Daniel Brake and Jonathan Hauenstein
For $\mathbb{k} = \mathbb{R}, \mathbb{C}$, take the **Puiseux series** over $\mathbb{k}$:

$$\mathbb{k}\{\{t\}\} = \bigcup_{n \in \mathbb{N}} \mathbb{k}(t^{1/n}).$$
For $k = \mathbb{R}, \mathbb{C}$, take the **Puiseux series** over $k$:

$$k\{\{t\}\} = \bigcup_{n \in \mathbb{N}} k((t^{1/n})).$$

This is an algebraically closed ($k = \mathbb{C}$) or real closed ($k = \mathbb{R}$) field with **valuation** $\text{val} : k\{\{t\}\}^* \to \mathbb{Q}$:

$$\text{val} \left( \sum c_q t^q \right) = \min \{q \mid c_q \neq 0\}.$$

This extends coordinate-wise to $\text{val} : k\{\{t\}\}^n \to \mathbb{Q}^n$. 
For $k = \mathbb{R}, \mathbb{C}$, take the **Puiseux series** over $k$:

$$k\{\{t\}\} = \bigcup_{n \in \mathbb{N}} k((t^{1/n})).$$

This is an algebraically closed ($k = \mathbb{C}$) or real closed ($k = \mathbb{R}$) field with **valuation** $\text{val} : k\{\{t\}\}^* \to \mathbb{Q}$:

$$\text{val} \left( \sum c_q t^q \right) = \min\{q \mid c_q \neq 0\}.$$

This extends coordinate-wise to $\text{val} : k\{\{t\}\}^n \to \mathbb{Q}^n$.

E.g. $\text{val}(3t^2 + 17t^5 + \ldots, 6t^{-1/3} + 5 + t^{1/3} + \ldots) = (2, -1/3)$. 

---

Cynthia Vinzant

Numerical methods for computing real and complex tropical curves
Puiseux series, valuations, and tropical varieties

For $\mathbb{k} = \mathbb{R}, \mathbb{C}$, take the Puiseux series over $\mathbb{k}$:

$$\mathbb{k}\{\{t\}\} = \bigcup_{n \in \mathbb{N}} \mathbb{k}(\{t^{1/n}\}).$$

This is an algebraically closed ($\mathbb{k} = \mathbb{C}$) or real closed ($\mathbb{k} = \mathbb{R}$) field with valuation $\text{val} : \mathbb{k}\{\{t\}\}^* \to \mathbb{Q}$:

$$\text{val}\left(\sum c_q t^q\right) = \min\{q \mid c_q \neq 0\}.$$

This extends coordinate-wise to $\text{val} : \mathbb{k}\{\{t\}\}^n \to \mathbb{Q}^n$.

E.g. $\text{val}(3t^2 + 17t^5 + \ldots, 6t^{-1/3} + 5 + t^{1/3} + \ldots) = (2, -1/3)$.

We can take the variety of $I \subset \mathbb{k}[x_1, \ldots, x_n]$ over $\mathbb{k}\{\{t\}\}$.

The $\mathbb{k}$-tropical variety of $I$ is $\text{Trop}_\mathbb{k}(I) = -\text{val}(\mathcal{V}_\mathbb{k}\{\{t\}\} I) \subset \mathbb{R}^n$. 
For $t \in \mathbb{R}_+$ and $V \subset \mathbb{k}^n$, consider the image under $\log_t(|\cdot|)$: 

$$A_t(I) = \log_t(|V|) \quad \text{(taken coordinate-wise)}.$$
Logarithmic limit sets

For $t \in \mathbb{R}^+$ and $V \subset \mathbb{R}^n$, consider the image under $\log_t(|\cdot|)$:

$$A_t(I) = \log_t(|V|) \quad \text{(taken coordinate-wise)}.$$

The **logarithmic limit set** of $V \subset \mathbb{R}^n$ is the limit as $t \to \infty$:

$$A_\infty(V) = \lim_{t \to \infty} A_t(V).$$
Logarithmic limit sets

For $t \in \mathbb{R}^+$ and $V \subset \mathbb{K}^n$, consider the image under $\log_t(|\cdot|)$:

$$\mathcal{A}_t(I) = \log_t(|V|) \quad \text{(taken coordinate-wise)}.$$ 

The logarithmic limit set of $V \subset \mathbb{K}^n$ is the limit as $t \to \infty$:

$$\mathcal{A}_\infty(V) = \lim_{t \to \infty} \mathcal{A}_t(V)$$

For both $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathcal{A}_\infty(\mathcal{V}_\mathbb{K}(I))$ equals $\text{Trop}_\mathbb{K}(I)$. 

\[ \begin{array}{c}
\text{Cynthia Vinzant} \\
\text{Numerical methods for computing real and complex tropical curves}
\end{array} \]
For \( w \in \mathbb{R}^n \), \( f = \sum_{\alpha} c_\alpha x^\alpha \), define \( \text{in}_w(f) = \sum_{\alpha \in A} c_\alpha x^\alpha \), where \( A \) is the set of \( \alpha \) maximizing \( w \cdot \alpha \). Then \( \text{in}_w(I) = \langle \text{in}_w(f) : f \in I \rangle \).
Connections with initial ideals

For \( w \in \mathbb{R}^n \), \( f = \sum_{\alpha} c_{\alpha} x^\alpha \), define \( \text{in}_w(f) = \sum_{\alpha \in \mathcal{A}} c_{\alpha} x^\alpha \), where \( \mathcal{A} \) is the set of \( \alpha \) maximizing \( w \cdot \alpha \). Then \( \text{in}_w(I) = \langle \text{in}_w(f) : f \in I \rangle \).

One can define \( \text{Trop}_\mathbb{C}(I) \) in terms of initial ideals:

\[
\text{Trop}_\mathbb{C}(I) = \{ w \in \mathbb{R}^n : \mathcal{V}(\text{in}_w(I)) \cap (\mathbb{C}^*)^n \neq \emptyset \}.
\]
Connections with initial ideals

For \( w \in \mathbb{R}^n \), \( f = \sum_\alpha c_\alpha x^\alpha \), define \( \text{in}_w(f) = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha \), where \( \mathcal{A} \) is the set of \( \alpha \) maximizing \( w \cdot \alpha \). Then \( \text{in}_w(I) = \left\langle \text{in}_w(f) : f \in I \right\rangle \).

One can define \( \text{Trop}_\mathbb{C}(I) \) in terms of initial ideals:
\[
\text{Trop}_\mathbb{C}(I) = \left\{ w \in \mathbb{R}^n : \mathcal{V}(\text{in}_w(I)) \cap (\mathbb{C}^*)^n \neq \emptyset \right\}.
\]

Over \( \mathbb{R} \), we only have that
\[
\text{Trop}_\mathbb{R}(I) \subseteq \left\{ w \in \mathbb{R}^n : \mathcal{V}(\text{in}_w(I)) \cap (\mathbb{R}^*)^n \neq \emptyset \right\}.
\]
Connections with initial ideals

For \( w \in \mathbb{R}^n \), \( f = \sum_{\alpha} c_{\alpha} x^{\alpha} \), define \( \text{in}_w(f) = \sum_{\alpha \in \mathcal{A}} c_{\alpha} x^{\alpha} \), where \( \mathcal{A} \) is the set of \( \alpha \) maximizing \( w \cdot \alpha \). Then \( \text{in}_w(I) = \langle \text{in}_w(f) : f \in I \rangle \).

One can define \( \text{Trop}_\mathbb{C}(I) \) in terms of initial ideals:
\[
\text{Trop}_\mathbb{C}(I) = \{ w \in \mathbb{R}^n : \mathcal{V}(\text{in}_w(I)) \cap (\mathbb{C}^*)^n \neq \emptyset \}.
\]

Over \( \mathbb{R} \), we only have that
\[
\text{Trop}_\mathbb{R}(I) \subseteq \{ w \in \mathbb{R}^n : \mathcal{V}(\text{in}_w(I)) \cap (\mathbb{R}^*)^n \neq \emptyset \}.
\]

E.g. For \( I = \langle (x - y)^2 + 1 \rangle \), \( \mathcal{V}_\mathbb{R}(I) = \emptyset \) and \( \text{Trop}_\mathbb{R}(I) = \emptyset \), but \( \mathcal{V}_\mathbb{R}^*(\text{in}_{(1,1)}(I)) = \mathcal{V}_\mathbb{R}^*((x - y)^2) \neq \emptyset \).
Example: sextic plane curve

Consider \( I = \langle x^6 - x^3 + y^2 \rangle \).
Example: sextic plane curve

Consider \( I = \langle x^6 - x^3 + y^2 \rangle \). Some points in \( \mathcal{V}_{\mathbb{C}\{\{t\}\}}(I) \) are …

\[
(x, y) = \left( t^2, t^3 - \frac{t^9}{2} - \frac{t^{15}}{8} + \ldots \right) \quad \rightarrow \quad -\text{val}(x, y) = (-2, -3)
\]
Example: sextic plane curve

Consider \( I = \langle x^6 - x^3 + y^2 \rangle \). Some points in \( V_{\mathbb{C}\{t\}}(I) \) are ...

\[(x, y) = \left( t^2, t^3 - \frac{t^9}{2} - \frac{t^{15}}{8} + \ldots \right) \quad \rightarrow \quad -\text{val}(x, y) = (-2, -3)\]

\[(x, y) = \left( 1 - \frac{t^2}{3} - \frac{4t^4}{9} + \ldots , t \right) \quad \rightarrow \quad -\text{val}(x, y) = (0, -1)\]
Example: sextic plane curve

Consider $I = \langle x^6 - x^3 + y^2 \rangle$. Some points in $V_{\mathbb{C}\{t\}}(I)$ are . . .

$$(x, y) = \left(t^2, t^3 - \frac{t^9}{2} - \frac{t^{15}}{8} + \ldots\right) \quad \rightarrow \quad -\text{val}(x, y) = (-2, -3)$$

$$(x, y) = \left(1 - \frac{t^2}{3} - \frac{4t^4}{9} + \ldots, t\right) \quad \rightarrow \quad -\text{val}(x, y) = (0, -1)$$

$$(x, y) = \left(\frac{1}{t}, -\frac{i}{t^3} + \frac{i}{2} + \frac{it^3}{8} + \ldots\right) \quad \rightarrow \quad -\text{val}(x, y) = (1, 3)$$

$\mathcal{V}_{\mathbb{R}^*}(I)$
Consider $I = \langle x^6 - x^3 + y^2 \rangle$. Some points in $\mathcal{V}_{\mathbb{C}^\{t\}}(I)$ are . . .

$(x, y) = \left(t^2, t^3 - \frac{t^9}{2} - \frac{t^{15}}{8} + \ldots\right) \rightarrow -\text{val}(x, y) = (-2, -3)$

$(x, y) = \left(1 - \frac{t^2}{3} - \frac{4t^4}{9} + \ldots, t\right) \rightarrow -\text{val}(x, y) = (0, -1)$

$(x, y) = \left(\frac{1}{t}, -\frac{i}{t^3} + \frac{i}{2} + \frac{it^3}{8} + \ldots\right) \rightarrow -\text{val}(x, y) = (1, 3)$
Tropical varieties are polyhedral fans

Theorem (Fundamental Theorem of Tropical Geometry)

For an irreducible ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$, $\text{Trop}_\mathbb{C}(I)$ is a rational pure-dimensional polyhedral fan of dimensional $d = \dim(\mathcal{V}_{\mathbb{C}^*}(I))$. Counting cones with appropriate multiplicities, it is balanced.
Tropical varieties are polyhedral fans

**Theorem (Fundamental Theorem of Tropical Geometry)**

For an irreducible ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$, $\text{Trop}_\mathbb{C}(I)$ is a rational pure-dimensional polyhedral fan of dimensional $d = \dim(\mathcal{V}_{\mathbb{C}^*}(I))$. Counting cones with appropriate multiplicities, it is balanced.

**Theorem (Alessandrini, 2013)**

For an ideal $I \subset \mathbb{R}[x_1, \ldots, x_n]$, $\text{Trop}_\mathbb{R}(I)$ is a rational polyhedral fan of dimensional $d \leq \dim(\mathcal{V}_{\mathbb{R}^*}(I))$. 
Symbolic Algorithm: \texttt{Gfan} – developed by Jensen
uses Gröbner bases to compute \( \text{Trop}_\mathbb{C}(I) \) for any \( I \subset \mathbb{Q}[x_1, \ldots, x_n] \).
Symbolic Algorithm: \texttt{Gfan} – developed by Jensen uses Gröbner bases to compute $\text{Trop}_C(I)$ for any $I \subset \mathbb{Q}[x_1, \ldots, x_n]$.

Numerical Algorithms:

- $\text{Trop}_C$ of hypersurfaces (Hauenstein, Sottile, 2014)
- $\text{Trop}_C$ of curves (Jensen, Leykin, Yu, 2015)
- $\text{Trop}_C$ and $\text{Trop}_R$ of curves (Brake, Hauenstein, V-)

Numerical methods for computing real and complex tropical curves
Symbolic Algorithm: \textsc{Gfan} – developed by Jensen uses Gröbner bases to compute $\text{Trop}_C(I)$ for any $I \subset \mathbb{Q}[x_1, \ldots, x_n]$.

Numerical Algorithms:
- $\text{Trop}_C$ of hypersurfaces (Hauenstein, Sottile, 2014)
- $\text{Trop}_C$ of curves (Jensen, Leykin, Yu, 2015)
- $\text{Trop}_C$ and $\text{Trop}_R$ of curves (Brake, Hauenstein, V-)

Numerical algorithms do not require defining equations.

Curves are tractable and are used in internal computations for $\text{Trop}_C$ of larger dimensional varieties.
Given an ideal $I \subseteq \mathbb{k}[x_1, \ldots, x_n]$ defining a curve $C = \mathcal{V}_\mathbb{k}(I)$, find intersection points $p$ of $C$ with $\{x_j = 0\}$, $j = 1, \ldots, n$.

For each $p$, numerically find an analytic parametrization of the branches of $C$ around $p$.

Calculate leading terms of the power series expansion of this parametrization using Cauchy integrals:

\[ f(z) \text{ analytic on } \{ z \in \mathbb{C} : |z| \leq \tau \}, \]

\[ f(k)(0) = \frac{k!}{2\pi i} \int_{\partial D(0,\tau)} f(\tau e^{i\theta}) (\tau e^{i\theta})^{k+1} \, d\theta, \]

and

\[ f(z) = \sum_{k=0}^{\infty} f(k)(0) \cdot \frac{1}{k!} \cdot z^k \text{ for } |z| \leq \tau. \]
Strategy for computing $\text{Trop}_k(I)$

Given an ideal $I \subset \mathbb{k}[x_1, \ldots, x_n]$ defining a curve $C = \mathcal{V}_k(I)$ . . .

- Find intersection points $p$ of $C$ with $\{x_j = 0\}_{j=1,\ldots,n}$.
Strategy for computing $\text{Trop}_k(I)$

Given an ideal $I \subset k[x_1, \ldots, x_n]$ defining a curve $C = \mathcal{V}_k(I) \ldots$

- Find intersection points $p$ of $C$ with $\{x_j = 0\}_{j=1,\ldots,n}$.
- For each $p$, numerically find an analytic parametrization of the branches of $C$ around $p$. 

Cauchy Integrals:

If $f(z)$ is analytic on $\{z \in \mathbb{C} : |z| \leq \tau\}$, then

$$f'(0) = \frac{1}{2\pi i} \int_{|z| = \tau} f(z) dz,$$

and

$$f(z) = \sum_{k=0}^{\infty} f^{(k)}(0) \cdot \frac{1}{k!} \cdot z^k$$

for $|z| \leq \tau$. 

Cynthia Vinzant
Numerical methods for computing real and complex tropical curves
Strategy for computing $\text{Trop}_{k}(I)$

Given an ideal $I \subset k[x_1, \ldots, x_n]$ defining a curve $C = V_k(I)$

- Find intersection points $p$ of $C$ with $\{x_j = 0\}_{j=1,\ldots,n}$.
- For each $p$, numerically find an analytic parametrization of the branches of $C$ around $p$.
- Calculate leading terms of the power series expansion of this parametrization using Cauchy integrals.

Cauchy Integrals:
If $f(z)$ is analytic on $\{z \in \mathbb{C}: |z| \leq \tau\}$, then
$$f(z) = \sum_{k=0}^{\infty} f^{(k)}(0) \cdot \frac{1}{k!} \cdot z^k$$
for $|z| \leq \tau$. 

Cynthia Vinzant
Numerical methods for computing real and complex tropical curves
Strategy for computing $\text{Trop}_k(I)$

Given an ideal $I \subset k[x_1, \ldots, x_n]$ defining a curve $C = \mathcal{V}_k(I)$ …

- Find intersection points $p$ of $C$ with $\{x_j = 0\}_{j=1,\ldots,n}$.
- For each $p$, numerically find an analytic parametrization of the branches of $C$ around $p$.
- Calculate leading terms of the power series expansion of this parametrization using Cauchy integrals.

**Cauchy Integrals:** If $f(z)$ is analytic on $\{z \in \mathbb{C} : |z| \leq \tau\}$, then

$$f^{(k)}(0) = \frac{k!}{2\pi} \int_0^{2\pi} \frac{f(\tau e^{i\theta})}{(\tau e^{i\theta})^{k+1}} d\theta,$$

and $f(z) = \sum_{k=0}^{\infty} f^{(k)}(0) \cdot \frac{1}{k!} \cdot z^k$ for $|z| \leq \tau$. 
Suppose \( \{P(s) : s \in [0, \tau]\} \subset C \) and \( P_j(s) = s \).
Suppose \( \{P(s) : s \in [0, \tau]\} \subset C \) and \( P_j(s) = s \).

Track the path \( P(\tau e^{i\theta}) \) for \( \theta \in [0, 2\pi] \).

The limit for \( \theta = 2\pi \) lies in \( C \cap \{x_j = \tau\} \).
Finding an analytic parametrization: monodromy

Suppose \( \{P(s) : s \in [0, \tau]\} \subset C \) and \( P_j(s) = s \).

Track the path \( P(\tau e^{i\theta}) \) for \( \theta \in [0, 2\pi] \).

The limit for \( \theta = 2\pi \) lies in \( C \cap \{x_j = \tau\} \).

The **cycle number** of the path is \( c \) if \( P(\tau e^{i\theta}) \) for \( \theta \in [0, c \cdot 2\pi] \) tracks to \( P(0) \).
Suppose \( \{P(s) : s \in [0, \tau]\} \subset C \) and \( P_j(s) = s \).

Track the path \( P(\tau e^{i\theta}) \) for \( \theta \in [0, 2\pi] \).

The limit for \( \theta = 2\pi \) lies in \( C \cap \{x_j = \tau\} \).

The **cycle number** of the path is \( c \) if \( P(\tau e^{i\theta}) \) for \( \theta \in [0, c \cdot 2\pi] \) tracks to \( P(0) \).

After re-parametrizing by \( s \mapsto s^c \), each coordinate \( P_k(s) \) analytic in \( s \).
Finding an analytic parametrization: monodromy

Suppose \( \{P(s) : s \in [0, \tau]\} \subset C \) and \( P_j(s) = s \).

Track the path \( P(\tau e^{i\theta}) \) for \( \theta \in [0, 2\pi] \).
The limit for \( \theta = 2\pi \) lies in \( C \cap \{x_j = \tau\} \).

The **cycle number** of the path is \( c \) if \( P(\tau e^{i\theta}) \) for \( \theta \in [0, c \cdot 2\pi] \) tracks to \( P(0) \).

After re-parametrizing by \( s \mapsto s^c \),
each coordinate \( P_k(s) \) analytic in \( s \).

Example: \( C = \mathcal{V}_\mathbb{C}(x^3 - y^2) \), \( P(s) = (s, s^{3/2}) \).
Cycle number = 2

\[ x = s^2 \]

\[ P(s) \]
Finding an analytic parametrization: monodromy

Suppose \( \{ P(s) : s \in [0, \tau] \} \subset C \) and \( P_j(s) = s \).

Track the path \( P(\tau e^{i\theta}) \) for \( \theta \in [0, 2\pi] \).

The limit for \( \theta = 2\pi \) lies in \( C \cap \{ x_j = \tau \} \).

The **cycle number** of the path is \( c \) if \( P(\tau e^{i\theta}) \) for \( \theta \in [0, c \cdot 2\pi] \) tracks to \( P(0) \).

After re-parametrizing by \( s \mapsto s^c \),

each coordinate \( P_k(s) \) analytic in \( s \).

Example: \( C = \mathcal{V}_\mathbb{C}(x^3 - y^2) \), \( P(s) = (s, s^{3/2}) \).

Cycle number = 2

Re-parametrize: \( s \mapsto s^2 \), \( P(s) = (s^2, s^3) \)
Let $C \subset \mathbb{C}^n$ be an irreducible curve.

- Take $\overline{C} \subset \mathbb{P}^n(\mathbb{C})$ and an affine slice $\widehat{C} = \{l = 1\} \cap \overline{C}$ containing all the points $\overline{C} \cap \mathcal{V}(x_0 x_1 \cdot x_n)$.
Let \( C \subset \mathbb{C}^n \) be an irreducible curve.

- Take \( \overline{C} \subset \mathbb{P}^n(\mathbb{C}) \) and an affine slice \( \hat{C} = \{ \ell = 1 \} \cap \overline{C} \) containing all the points \( \overline{C} \cap \mathcal{V}(x_0 x_1 \cdots x_n) \).
- Compute \( T^* = \min(|T|) \) such that \( \{ x_j - T \} \) is tangent to \( \hat{C} \).
Sketch of Algorithm

Let $C \subset \mathbb{C}^n$ be an irreducible curve.

- Take $\overline{C} \subset \mathbb{P}^n(\mathbb{C})$ and an affine slice $\hat{C} = \{\ell = 1\} \cap \overline{C}$ containing all the points $\overline{C} \cap \mathcal{V}(x_0x_1 \cdot x_n)$.
- Compute $T^* = \min(|T|)$ such that $\{x_j - T\}$ is tangent to $\hat{C}$.
- For $\tau < T^*$, slice $\hat{C}$ with $\{x_j - \tau\}$.
Sketch of Algorithm

Let $C \subset \mathbb{C}^n$ be an irreducible curve.

- Take $\overline{C} \subset \mathbb{P}^n(\mathbb{C})$ and an affine slice $\hat{C} = \{ \ell = 1 \} \cap \overline{C}$ containing all the points $\overline{C} \cap \mathcal{V}(x_0x_1 \cdot x_n)$.

- Compute $T^* = \min(|T|)$ such that $\{x_j - T\}$ is tangent to $\hat{C}$.

- For $\tau < T^*$, slice $\hat{C}$ with $\{x_j - \tau\}$.

- Track points to $\{x_j = 0\}$ and compute valuations using re-parametrization and Cauchy integrals.
Let $C \subset \mathbb{C}^n$ be an irreducible curve.

- Take $\overline{C} \subset \mathbb{P}^n(\mathbb{C})$ and an affine slice $\widehat{C} = \{ \ell = 1 \} \cap \overline{C}$ containing all the points $\overline{C} \cap \mathcal{V}(x_0x_1 \cdot x_n)$.
- Compute $T^* = \min(|T|)$ such that $\{x_j - T\}$ is tangent to $\widehat{C}$.
- For $\tau < T^*$, slice $\widehat{C}$ with $\{x_j - \tau\}$.
- Track points to $\{x_j = 0\}$ and compute valuations using re-parametrization and Cauchy integrals.
Replace $C = \mathcal{V}(x_1^3x_2 - x_1x_2^3 + x_1^3 - x_2^3) \subset \mathbb{C}^2$ with
\[\tilde{C} = \mathcal{V}(x_1^3x_2 - x_1x_2^3 + x_0x_1^3 - x_0^2x_2^2, \ x_0 + x_1 + 2x_2 - 1) \subset \mathbb{C}^3.\]
Example: quartic plane curve

Replace \( C = \mathcal{V}(x_1^3x_2 - x_1x_2^3 + x_1^3 - x_2^2) \subset \mathbb{C}^2 \) with

\( \hat{C} = \mathcal{V}(x_1^3x_2 - x_1x_2^3 + x_0x_1^3 - x_0^2x_2^2, \ x_0 + x_1 + 2x_2 - 1) \subset \mathbb{C}^3. \)

\( \hat{C} \cap \mathcal{V}(x_0x_1x_2) = \{(0, 1, 0), (0, 1/3, 1/3), (0, 0, 1/2), (0, -1, 1), (1, 0, 0)\} \)
Example: quartic plane curve

Replace \( C = \mathcal{V}(x_1^3x_2 - x_1x_2^3 + x_1^3 - x_2^2) \subset \mathbb{C}^2 \) with
\[
\hat{C} = \mathcal{V}(x_1^3x_2 - x_1x_2^3 + x_0x_1^3 - x_0^2x_2^2, \ x_0 + x_1 + 2x_2 - 1) \subset \mathbb{C}^3.
\]
\[
\hat{C} \cap \mathcal{V}(x_0x_1x_2) = \{(0, 1, 0), (0, 1/3, 1/3), (0, 0, 1/2), (0, -1, 1), (1, 0, 0)\}
\]
The point \( p \approx (0.8293, 0.1, 0.0354) \subset \hat{C} \cap \{x_1 = .1\} \) tracks to \((1, 0, 0)\).
Example: quartic plane curve

Replace $C = \mathcal{V}(x_1^3x_2 - x_1x_2^3 + x_1^3 - x_2^2) \subset \mathbb{C}^2$ with

$\hat{C} = \mathcal{V}(x_1^3x_2 - x_1x_2^3 + x_0x_1^3 - x_0^2x_2^2, x_0 + x_1 + 2x_2 - 1) \subset \mathbb{C}^3$.

$\hat{C} \cap \mathcal{V}(x_0x_1x_2) = \{(0, 1, 0), (0, 1/3, 1/3), (0, 0, 1/2), (0, -1, 1), (1, 0, 0)\}$

The point $p \approx (0.8293, 0.1, 0.0354) \subset \hat{C} \cap \{x_1 = .1\}$ tracks to $(1, 0, 0)$.

Valuation of the path is $(0, 2, 3) \rightarrow (-2, -3) \in \text{Trop}_{\mathbb{C}}(C)$.
Real Tropical Strategy

We can compute $\text{Trop}_\mathbb{R}(C)$ similarly to $\text{Trop}_\mathbb{C}(C)$. This requires checking $\{x_j = \pm \tau\}$ for real points and only considering real paths converging to $\hat{C} \cap \{x_j = 0\}$.
We can compute $Trop_R(C)$ similarly to $Trop_C(C)$. This requires checking $\{x_j = \pm \tau\}$ for real points and only considering real paths converging to $\hat{C} \cap \{x_j = 0\}$.

Keeping track of signs of the parametrized paths gives the signed real tropical variety.
A central curve defined by ...

\[
\begin{bmatrix}
  x_1s_1 - x_j s_j & \text{for } j = 2, \ldots, 7 \\
  -u_0 + t^2 - x_1 \\
  -v_0 + t^4 - x_2 \\
  u_1 - x_3 \\
  v_1 - x_4 \\
  t(u_0 + v_0) - v_1 - x_5 \\
  t^2u_0 - u_1 - x_6 \\
  t^2v_0 - u_1 - x_7 \\
  t^2x_2 + x_3 + x_7 - t^6 \\
  t^2x_1 + x_3 + x_6 - t^4 \\
  tx_1 + tx_2 + x_4 + x_5 - t^3 - t^5 \\
  s_1 - ts_5 - t^2s_6 \\
  s_2 - ts_5 - t^2s_7 - 1 \\
  s_3 - s_6 - s_7 \\
  s_4 - s_5
\end{bmatrix}
\]

Allamigeon, Benchimol, Gaubert, and Joswig use real tropical methods to construct a family of linear programs whose central curves have high total curvature.
Computing curves in large spaces

A central curve defined by . . .

\[
\begin{bmatrix}
x_1 s_1 - x_j s_j & \text{for } j = 2, \ldots, 7 \\
-u_0 + t^2 - x_1 \\
-v_0 + t^4 - x_2 \\
u_1 - x_3 \\
v_1 - x_4 \\
t(u_0 + v_0) - v_1 - x_5 \\
t^2 u_0 - u_1 - x_6 \\
t^2 v_0 - u_1 - x_7 \\
t^2 x_2 + x_3 + x_7 - t^6 \\
t^2 x_1 + x_3 + x_6 - t^4 \\
t x_1 + t x_2 + x_4 + x_5 - t^3 - t^5 \\
s_1 - t s_5 - t^2 s_6 \\
s_2 - t s_5 - t^2 s_7 - 1 \\
s_3 - s_6 - s_7 \\
s_4 - s_5
\end{bmatrix}
\]

Allamigeon, Benchimol, Gaubert, and Joswig use real tropical methods to construct a family of linear programs whose central curves have high total curvature.

In this example, the polynomials define a reducible algebraic variety consisting of two 3-planes, five 2-planes, four lines, and a degree 10 central curve \( C \).
Computing curves in large spaces

A central curve defined by ...

\[
\begin{bmatrix}
  x_1s_1 - x_2s_2 & \cdots & x_1s_1 - x_7s_7 \\
  -u_0 + t^2 - x_1 \\
  -v_0 + t^4 - x_2 \\
  u_1 - x_3 \\
  v_1 - x_4 \\
  t(u_0 + v_0) - v_1 - x_5 \\
  t^2u_0 - u_1 - x_6 \\
  t^2v_0 - u_1 - x_7 \\
  t^2x_2 + x_3 + x_7 - t^6 \\
  t^2x_1 + x_3 + x_6 - t^4 \\
  tx_1 + tx_2 + x_4 + x_5 - t^3 - t^5 \\
  s_1 - ts_5 - t^2s_6 \\
  s_2 - ts_5 - t^2s_7 - 1 \\
  s_3 - s_6 - s_7 \\
  s_4 - s_5 
\end{bmatrix}
\]

Allamigeon, Benchimol, Gaubert, and Joswig use real tropical methods to construct a family of linear programs whose central curves have high total curvature.

In this example, the polynomials define a reducible algebraic variety consisting of two 3-planes, five 2-planes, four lines, and a degree 10 central curve \( C \).

Our algorithm find that the tropical variety \( \text{Trop}_\mathbb{C}(C) = \text{Trop}_\mathbb{R}(C) \) consists of 10 rays with multiplicites 6, 4, 3, 2, 2, 1, 1, 1, 1, 1.
Numerical methods can be used to extract combinatorial data from complex and real algebraic varieties, including *tropicalizations*.
Numerical methods can be used to extract combinatorial data from complex and real algebraic varieties, including *tropicalizations*.

We would like to develop these methods for varieties of dimension $\geq 1$, like *surfaces*, over both $\mathbb{C}$ and $\mathbb{R}$.

Tropicalizations of varieties higher dimension can also be used to compute tropical varieties of ideals in $\mathbb{K}\{t\}[x_1, \ldots, x_n]$, which are used in many applications (like central curves).
Numerical methods can be used to extract combinatorial data from complex and real algebraic varieties, including tropicalizations.

We would like to develop these methods for varieties of dimension $\geq 1$, like surfaces, over both $\mathbb{C}$ and $\mathbb{R}$.

Tropicalizations of varieties higher dimension can also be used to compute tropical varieties of ideals in $\mathbb{K}\{t\}[x_1, \ldots, x_n]$, which are used in many applications (like central curves).