

INDEPENDENCE POSETS

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Triangle Lectures in Combinatorics

November 14, 2020

	Paid Only	Breakout Rooms	Video	Antitrust
MS Teams				
Zoom				
Google Meet				
US Postal Service				

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Zoom		✓	✓	
Google Meet			✓	✓
US Postal Service	✓			

General: a set of **objects** \mathcal{J} and **properties** \mathcal{M} , with maps

$$m : \mathcal{J} \rightarrow 2^{\mathcal{M}} \quad (\text{an object } \mapsto \text{its properties})$$

$$j : \mathcal{M} \rightarrow 2^{\mathcal{J}} \quad (\text{a property } \mapsto \text{objects with that property})$$

RUDOLF WILLE'S FORMAL CONCEPT ANALYSIS

subsets of **objects** \mapsto shared **properties**
subsets of **properties** \mapsto common **objects**

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Zoom		✓	✓	
Google Meet			✓	✓
US Postal Service	✓			

- the common properties of **MS Teams**, **Google Meet** are

$$m \left(\left\{ \text{MS Teams}, \text{Google Meet} \right\} \right) = \left\{ \text{Video}, \text{Antitrust} \right\}.$$

- the objects with both **Video**, **Antitrust** are

$$j \left(\left\{ \text{Video}, \text{Antitrust} \right\} \right) = \left\{ \text{MS Teams}, \text{Google Meet} \right\}.$$

- so the pair $\left(\left\{ \text{MS Teams}, \text{Google Meet} \right\}, \left\{ \text{Video}, \text{Antitrust} \right\} \right)$ is the idea **“big corporation video conferencing software”**

General: a *concept* is a pair $(X, Y) \in 2^{\mathcal{J}} \times 2^{\mathcal{M}}$ such that:

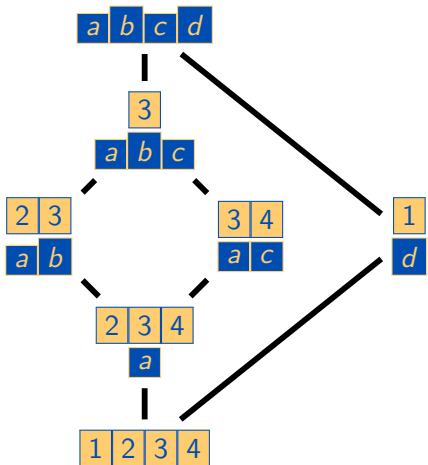
$$m(X) = \bigcap_{x \in X} m(x) = Y$$

$$j(Y) = \bigcap_{y \in Y} j(y) = X.$$

The *poset of concepts* is the set of all concepts ordered by

$$\left(\begin{array}{c} X \\ Y \end{array} \right) \leq \left(\begin{array}{c} X' \\ Y' \end{array} \right) \text{ iff } X \subseteq X'.$$

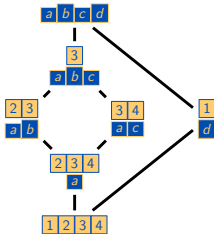
	1	2	3	4
a		✓	✓	✓
b		✓	✓	
c			✓	✓
d	✓			



The *poset of concepts* is the set of concepts ordered by

$$\left(\begin{array}{|c|} \hline X \\ \hline \end{array}, \begin{array}{|c|} \hline Y \\ \hline \end{array} \right) \leq \left(\begin{array}{|c|} \hline X' \\ \hline \end{array}, \begin{array}{|c|} \hline Y' \\ \hline \end{array} \right) \text{ iff}$$

$$\begin{array}{|c|} \hline X \\ \hline \end{array} \subseteq \begin{array}{|c|} \hline X' \\ \hline \end{array} \text{ or (equivalently) } \begin{array}{|c|} \hline Y' \\ \hline \end{array} \subseteq \begin{array}{|c|} \hline Y \\ \hline \end{array}.$$



THEOREM (LATTICE REPRESENTATION I)

*The poset of concepts is a **lattice**, and every **lattice** arises as a poset of concepts.*

LATTICES

“Never in the history of mathematics has a mathematical theory been the object of such vociferous vituperation as lattice theory. Dedekind, Jónsson, Kurosh, Malcev, Ore, von Neumann, Tarski, and most prominently Garrett Birkhoff have contributed a new vision of mathematics, a vision that has been cursed by a conjunction of misunderstandings, resentment, and raw prejudice.”

—Gian-Carlo Rota (*The Many Lives of Lattice Theory*)

DEFINITION

A (finite) poset L is a *lattice* if each pair $x, y \in L$ have:

a *join* (least upper bound): $x \vee y$, and

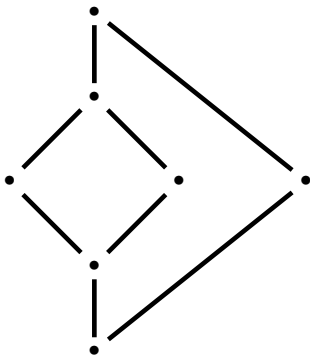
a *meet* (greatest lower bound): $x \wedge y$.

“Like its elder sister group theory, lattice theory is a fruitful source of abstract concepts, common to traditionally unrelated branches of mathematics. Both subjects are based on postulates of an extremely simple and general nature. It was this which convinced me from the first that lattice theory was destined to play—indeed, already did play implicitly—a fundamental role in mathematics. Though its importance will probably never equal that of group theory, I do believe that it will achieve a comparable status.”

—Garrett Birkhoff (*Lattice Theory*, Second Edition).

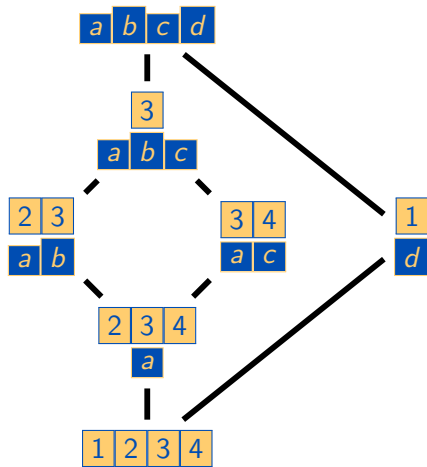
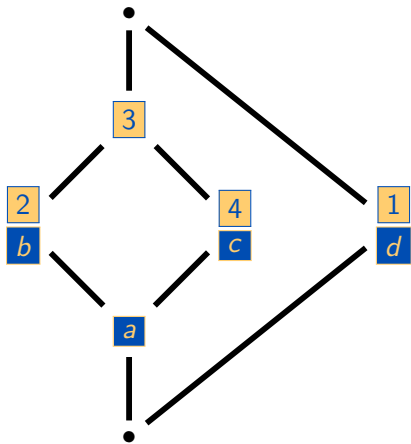
DEFINITION

- ▶ An element $j \in L$ is *join-irreducible* if when $j = \bigwedge S$, then $j \in S$. Write \mathcal{J} for the set of all join-irreducibles.
- ▶ An element $m \in L$ is *meet-irreducible* if when $m = \bigvee S$, then $m \in S$. Write \mathcal{M} for the set of all meet-irreducibles.



For $x \in L$, write $\mathcal{J}(x) = \{j \in \mathcal{J} : j \leq x\}$

$\mathcal{M}(x) = \{m \in \mathcal{M} : m \geq x\}$.



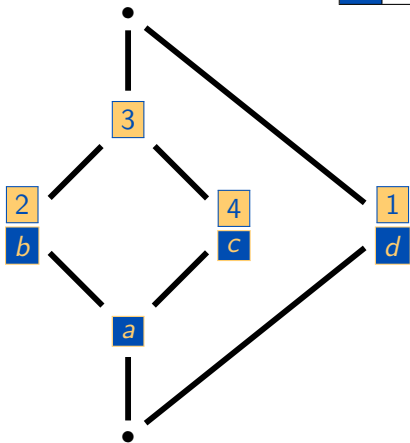
Since any element is the join of the join-irrs below it and the meet of the meet-irrs above it:

THEOREM (LATTICE REPRESENTATION II)

Every finite lattice L is isomorphic to (both)

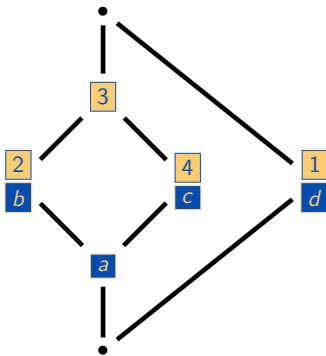
- ▶ $\{\mathcal{J}(x) : x \in L\}$ under inclusion, and
- ▶ $\{\mathcal{M}(x) : x \in L\}$ under reverse inclusion.

	1	2	3	4
a				
b				
c				
d				



Show $j \leq m$

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MS Teams		✓	✓	✓
Zoom		✓	✓	
Google Meet			✓	✓
US Postal Service	✓			



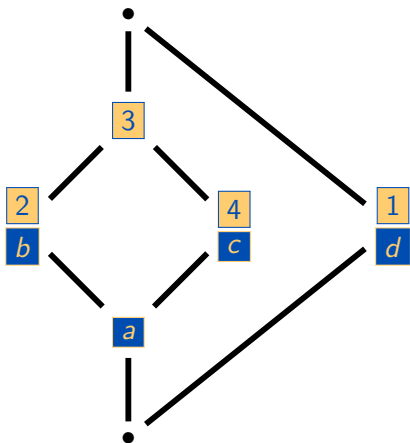
HISTORICAL NOTES

- ▶ *Formal concept analysis* was Rudolf Wille's "restructuring" of lattice theory (published 1982).
- ▶ Previously appeared in the 1973 thesis of George Markowsky as the *poset of irreducibles*.
- ▶ First (?) appeared in 1965 work of Marc Barbut as *l'algèbre des techniques d'analyse hiérarchique*.
- ▶ Similar ideas due to M. P. Schützenberger.

GEORGE MARKOWSKY'S EXTREMAL LATTICES

“When do lattices have a ‘compact’ representation theorem?”

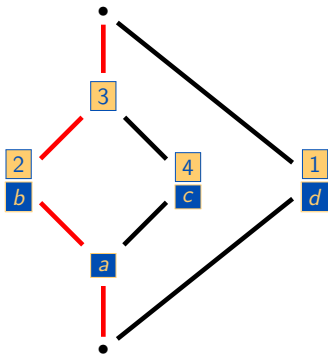
	1	2	3	4
a		✓	✓	✓
b		✓	✓	
c			✓	✓
d	✓			



DEFINITION (MARKOWSKY)

A lattice is *extremal* if $|\mathcal{J}| = |\mathcal{M}| = \text{length } L$.

	1	2	3	4
a		✓	✓	✓
b		✓	✓	
c			✓	✓
d	✓			



Key point: A chain of maximal length $\hat{0} = x_0 \triangleleft x_1 \triangleleft \dots \triangleleft x_n = \hat{1}$ lets us pair up \mathcal{J} and \mathcal{M} by $j_i \leftrightarrow m_i$, where:

$$\begin{aligned}
 x_i &= j_1 \vee \dots \vee j_i \\
 &= m_{i+1} \wedge \dots \wedge m_n
 \end{aligned}$$

A chain of maximal length lets us pair up \mathcal{J} and \mathcal{M} :

$$a \leftrightarrow 1, b \leftrightarrow 4, c \leftrightarrow 2, d \leftrightarrow 3$$

	1	2	3	4
a	X	✓	✓	✓
b	X	✓	✓	X
c	X	X	✓	✓
d	✓	X	X	X

\mapsto

1 4 2 3

\mapsto

a b c d

d 3

b 4

c 2

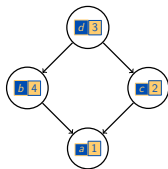
a 1

THEOREM (MARKOWSKY)

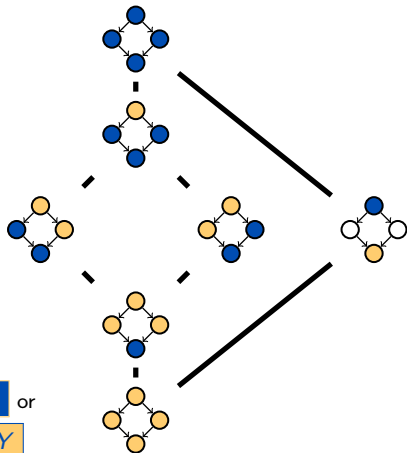
Every finite extremal lattice L is isomorphic to the lattice of maximal orthogonal pairs of a finite directed acyclic graph G .

$$\left\{ (X, Y) : \begin{array}{l} X \cap Y = \emptyset \\ \text{no arrow } X \rightarrow Y \\ X, Y \text{ maximal} \end{array} \right\}$$

ordered by



$$(X, Y) \leq (X', Y') \text{ iff } \begin{array}{l} X \subseteq X' \text{ or} \\ Y' \subseteq Y \end{array}$$

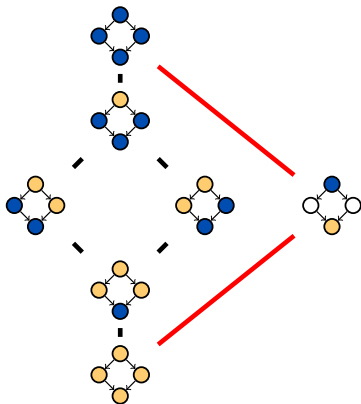


THEOREM (MARKOWSKY)

A finite **graded** extremal lattice L is distributive, in which case G is the comparability graph of a poset.

COROLLARY: Birkhoff's theorem for distributive lattices.

	1	2	3	4
a		✓	✓	✓
b		✓	✓	
c			✓	✓
d	✓			

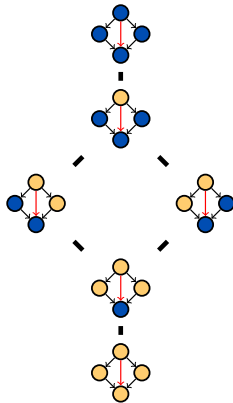


THEOREM (MARKOWSKY)

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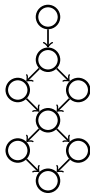
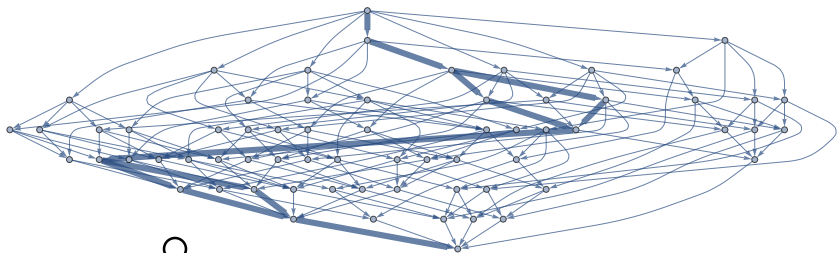
COROLLARY: Birkhoff's theorem for distributive lattices.

	1	2	3	4
a		✓	✓	✓
b		✓	✓	
c			✓	✓
d	✗			



THEOREM (MARKOWSKY)

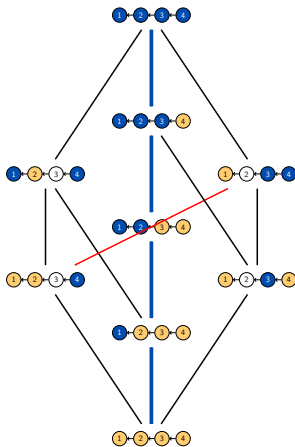
A finite graded extremal lattice L is distributive, in which case G is the comparability graph of a poset.



HUGH THOMAS'S TRIM LATTICES

“What if a distributive lattice weren't graded?”

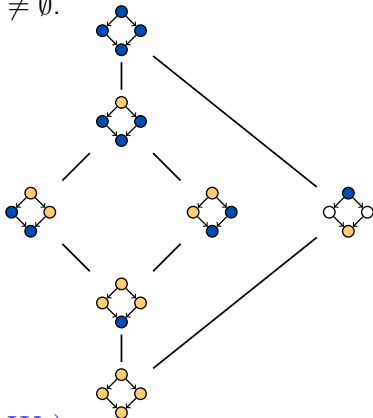
Call a relation $(X, Y) \leq (X', Y')$ in an extremal lattice *overlapping* if $Y \cap X' \neq \emptyset$.



THEOREM (THOMAS-W.)

An extremal lattice is *trim* iff every relation is overlapping iff every cover relation is overlapping.

Call a relation $(\boxed{X}, \boxed{Y}) \leq (\boxed{X'}, \boxed{Y'})$ in an extremal lattice *overlapping* if $\boxed{Y} \cap \boxed{X'} \neq \emptyset$.



THEOREM (THOMAS-W.)

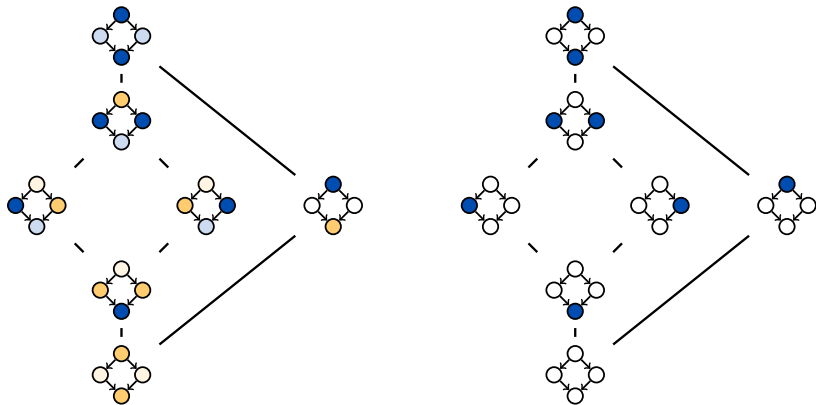
An extremal lattice is *trim* iff every relation is overlapping iff every cover relation is overlapping. This lets us label every cover relation.

THEOREM (THOMAS-W.)

$$\mathcal{D} : x \mapsto \{g \in G : y \triangleleft x\}$$

$$\mathcal{U} : x \mapsto \{g \in G : x \triangleleft y\}$$

are each bijections from the elements of a trim lattice L to the independent sets of G .



INDEPENDENCE POSETS

“What if a distributive lattice weren't a lattice?”

[Click for interactive version](#)

Recall: For trim lattices L ,

- ▶ Every cover relation is overlapping ($\boxed{Y} \cap \boxed{X'} \neq \emptyset$) and
- ▶ There are bijections

$$\boxed{\mathcal{D}} : x \mapsto \{g \in G : y \leq x\}$$

$$\boxed{\mathcal{U}} : x \mapsto \{g \in G : x \leq y\}$$

from L to the independent sets of G .

Questions:

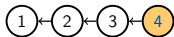
- ▶ How do $\boxed{\mathcal{D}}$ and $\boxed{\mathcal{U}}$ fit together?
- ▶ Can we simply express cover relations on $\boxed{\mathcal{D}}$ and $\boxed{\mathcal{U}}$?

Fix G a finite directed acyclic graph.

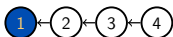
DEFINITION (THOMAS-W.)

A pair $(\mathcal{D}, \mathcal{U})$ of independent sets of G is *orthogonal* if $\mathcal{D} \cap \mathcal{U} = \emptyset$ and there is no arrow $\mathcal{D} \rightarrow \mathcal{U}$. An orthogonal pair is *tight* if whenever

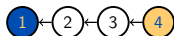
▶ an element of \mathcal{D} is increased



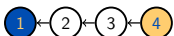
▶ an element of \mathcal{U} is decreased



▶ an element is added to \mathcal{D}



▶ an element is added to \mathcal{U}



the result isn't orthogonal.

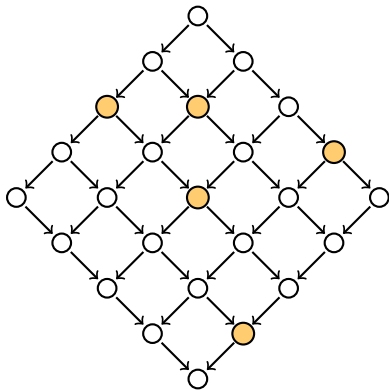
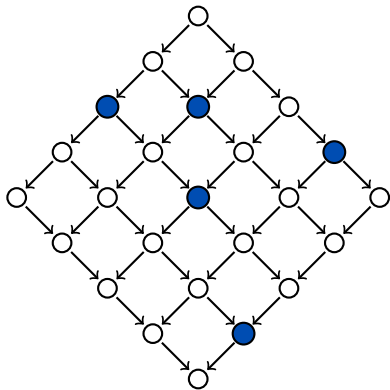
THEOREM (THOMAS-W.)

Any independent set \mathcal{I} can be uniquely completed to

a top $(\mathcal{I}, \mathcal{U})$

and

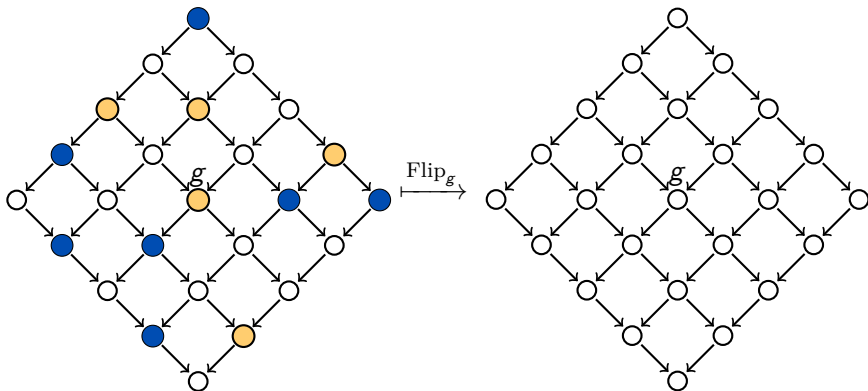
a top $(\mathcal{D}, \mathcal{I})$.



DEFINITION (“FLIPS”)

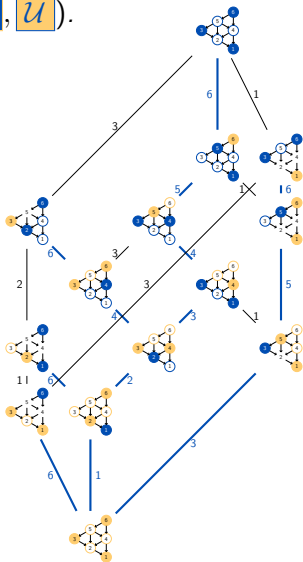
If $g \in \mathcal{D}$ or $g \in \mathcal{U}$, $\text{flip}_g(\mathcal{D}, \mathcal{U})$ is defined by:

- (I) fix all elements of \mathcal{D} not $< g$
- (II) fix all elements of \mathcal{U} not $> g$
- (III) swap g from \mathcal{D} to \mathcal{U} or vice-versa
- (IV) complete \mathcal{D} and \mathcal{U} (uniquely).



THEOREM (THOMAS-W.)

The *independence poset* of G is the poset with cover relations $(\mathcal{D}, \mathcal{U}) \prec \text{flip}_g(\mathcal{D}, \mathcal{U})$.



THEOREM (THOMAS-W.)

If an independence poset is a lattice, it is a trim lattice. If it is further a graded lattice, it is a distributive lattice.

“What if a distributive lattice weren’t a lattice?”

Thank You!

FUTURE WORK:

- ▶ Many(!) combinatorial objects can be encoded as independent sets. What new and old structures arise using independence posets?
- ▶ With generalized rowmotion, new examples for DAC?
- ▶ Can define posets on integer points in dilations of “independence polytopes”. And?
- ▶ Random sampling (CFTP)?