Interlacing methods in Extremal Combinatorics

Hao Huang

Emory University

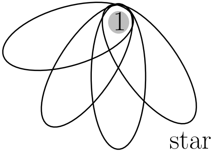
Nov 14, 2020

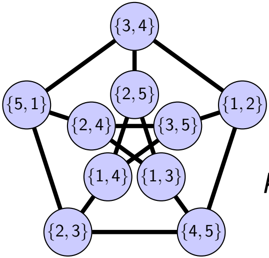
The Erdős-Ko-Rado (EKR) Theorem

Theorem (Erdős, Ko, Rado 1961)

For $n \geq 2k$, an intersecting family $\mathcal{F}$ of k-sets of $[n]$ has size at most $\binom{n-1}{k-1}$.

$\max|\mathcal{F}| = \alpha(KG(n,k))$.
$KG(n,k)$ is the Kneser graph whose vertices are all the k-sets and edges are disjoint pairs.





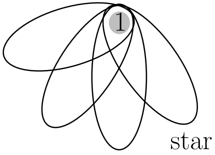
$KG(5,2)$

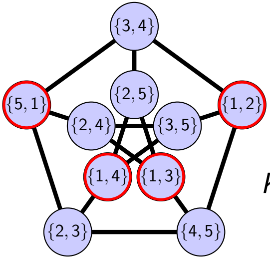
The Erdős-Ko-Rado (EKR) Theorem

Theorem (Erdős, Ko, Rado 1961)

For $n \geq 2k$, an intersecting family $\mathcal{F}$ of k-sets of $[n]$ has size at most $\binom{n-1}{k-1}$.

$\max |\mathcal{F}| = \alpha(KG(n,k))$.
$KG(n,k)$ is the Kneser graph whose vertices are all the k-sets and edges are disjoint pairs.





$KG(5,2)$

Eigenvalue Interlacing and independence number

Cauchy's Interlace Theorem

Let A be a **symmetric** matrix of size n, and B is a principal submatrix of A of size $m \leq n$. Suppose the eigenvalues of A are

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

and the eigenvalues of B are

$$\mu_1 \geq \cdots \geq \mu_m.$$

Then for $1 \leq i \leq m$, we have

$$\lambda_{i+n-m} \leq \mu_i \leq \lambda_i.$$

It follows from the Courant–Fischer–Weyl min-max principle.

Bounds on graph independence number

The inertia bound

The independence number $\alpha(G)$ of a graph G satisfies

$$\alpha(G) \leq \min\{n_{\geq 0}(A_G), n_{\leq 0}(A_G)\}.$$

Proof.

A slightly more sophisticated use of interlacing gives

The ratio bound

The independence number $\alpha(G)$ of a d-regular n-vertex graph G satisfies

$$\alpha(G) \leq n \cdot \frac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}.$$

Algebraic Proof of the Erdős–Ko–Rado Theorem

The eigenvalues of Kneser graph $KG(n,k)$ are

$$\lambda_j = (-1)^j \binom{n-k-j}{k-j}, \text{ with multiplicity } m_j = \binom{n}{j} - \binom{n}{j-1}.$$

Ratio bound

$$\alpha(KG(n,k)) \le \binom{n}{k} \cdot \frac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}} = \binom{n-1}{k-1}.$$

Inetia bound

$$\alpha(KG(n,k)) \le \min\left\{ \sum_{j \text{ odd}} m_j, \sum_{j \text{ even}} m_j \right\} = \binom{n-1}{k-1}.$$

Degree EKR Theorem

Theorem (H., Zhao 2017)

For $n \geq 2k+1$ and an intersecting family $\mathcal{F}$ of k-subsets of $[n]$, there exists an element $i \in [n]$ contained in at most $\binom{n-2}{k-2}$ subsets of $\mathcal{F}$.

Our degree EKR Theorem implies the EKR Theorem.

Question 1

Is it true that for $n \geq 2k+1$ (or $n \geq 2k+c$), and every intersecting family of k-sets of $[n]$, there exists a subset S of size d contained in at most $\binom{n-d-1}{k-d-1}$ subsets of $\mathcal{F}$?

Open even for $d = 2$. True for $n \geq 2k + 3d/(1 - d/k)$ (Kupavskii).

Question 2

Are there spectral proofs of the Hilton–Milner and Complete Intersection theorems?

The isoperimetric inequality

The isoperimatric inequality

The area of any region in the plane bounded by a curve of a fixed length can never exceed the area of a circle whose boundary has that length, i.e. $A \le L^2/(4\pi)$.

Dido Purchases Land for the Foundation of Carthage. Engraving by Matthäus Merian the Elder, in *Historische Chronica*, Frankfurt a.M., 1630. Dido's people cut the hide of an ox into thin strips and try to enclose a maximal domain.

The isodiametric inequality

There is a slightly less well-known isodiametric inequality.

The diameter of a set S is defined as the maximum distance between two points of S.

The isodiametric inequality

In $\mathbb{R}^n$, suppose S is a compact set with diameter $\text{diam}(S)$ and volume $\text{vol}(S)$, then

$$\text{vol}(S) \leq \text{vol}(B_1) \cdot (\text{diam}(S)/2)^n,$$

and the equality holds if and only if S is a ball.

Follows from Steiner symmetrization + Brunn–Minkowski.

Isodiametric inequality on discrete hypercubes

Kleitman's Theorem

Suppose $\mathcal{F}$ is a collection of binary vectors in $\{0,1\}^n$, such that the Hamming distance between any two vectors is at most $d < n$. Then

$$|\mathcal{F}| \le \text{size of the largest Hamming ball of radius } d/2.$$

Kleitman used somewhat complicated combinatorial shifting techniques.

Theorem (H., Klurman, Pohoata 2019)

Kleitman's Theorem follows from the inertia bound when applied to a carefully chosen pseudo-adjacency matrix.

Our method is also widely applicable to other allowed distance sets.

Other distance problems on hypercubes

The orthogonality graph Ω_n has $V(\Omega_n) = \{-1, 1\}^n$, two vectors are adjacent if they are orthogonal.

Conjecture (Galliard 2001)

For $n = 4k$,

$$\alpha(\Omega_n) = 4\left(\binom{n-1}{0} + \cdots + \binom{n-1}{n/4-1}\right).$$

Known for $n = 4p^k$ (Frankl 1986 for odd p, Ihringer–Tanaka 2019 for $p = 2$).

Problem

Find an analogue of the inertia bound over finite fields/rings.

The Sensitivity Conjecture (I)

A Boolean function takes the form $f : \{0,1\}^n \to \{0,1\}$.

- f can be expressed as a unique multi-linear real polynomial.
- Not every multilinear real polynomial gives a Boolean function.

Definition

Given a boolean function $f : \{0,1\}^n \to \{0,1\}$.

- The local sensitivity $\mathrm{s}(f,x)$ on the input x is defined as the number of indices i, such that $f(x) \neq f(x^{\{i\}})$.
- The sensitivity $\mathrm{s}(f)$ of f is $\max_{x\in\{0,1\}^n} \mathrm{s}(f,x)$.
- The degree $\deg(f)$ is the degree of f as a real multilinear polynomial.

e.g. $\mathrm{s}(OR_n) = n$, attained by the all-zero vector. $\deg(OR_n) = n$, since $OR_n = 1 - (1-x_1)(1-x_2)\cdots(1-x_n)$.

The Sensitivity Conjecture (II)

Sensitivity Conjecture (Nisan, Szegedy 1992)

For every boolean function f,

$$\deg(f) \le \text{poly}(\text{s}(f)).$$

Block sensitivity $\text{bs}(f)$	polynomially related
Decision tree complexity $D(f)$	
Certificate complexity $C(f)$	
Degree (as real polynomial) $\deg(f)$	
Approximate degree $\widetilde{\deg}(f)$	
Randomized query complexity $R(f)$	
Quantum query complexity $Q(f)$	

Sensitivity Conjecture $\Longrightarrow$ *"Sensitivity would cease to be an outlier and joins a large and happy flock."*

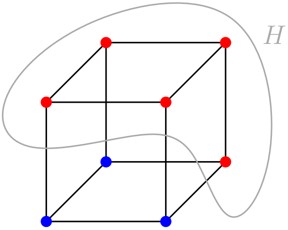
– Scott Aaronson

The Gotsman–Linial equivalence

Theorem (Gotsman, Linial 1992)

The following are equivalent for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

- For any induced subgraph H of Q^n with $|V(H)| \neq 2^{n-1}$, we have
$$\max\{\Delta(H), \Delta(Q^n - H)\} \geq h(n).$$
- For any boolean function f, we have $s(f) \geq h(\deg(f))$.



The Sensitivity Theorem

Theorem (H. 2019)

Every $(2^{n-1}+1)$-vertex induced subgraph of Q^n contains a vertex of degree at least $\sqrt{n}$.

Previously the best lower bound was $(1/2 - o(1)) \log_2 n$, by Chung, Füredi, Graham, Seymour in 1988.

Corollary

For every boolean function f,

$$\mathrm{s}(f) \geq \sqrt{\deg(f)},$$

and thus the Sensitivity Conjecture is true.

This bound is sharp by the AND-of-OR function $\bigwedge_i (\bigvee_j x_{ij})$.

Idea 1. Consider the largest eigenvalue.

Lemma

For every graph G with largest eigenvalue λ_1,

$$\sqrt{\Delta(G)} \le \lambda_1 \le \Delta(G).$$

Proof. For the upper bound, let $\vec{v}$ be an eigenvector of λ_1, and v_i is its coordinate largest in absolute value, then

$$|\lambda_1 v_i| = |(A_G \vec{v})_i| = |\sum_{j \sim i} v_j| \le \Delta(G) \cdot |v_i|,$$

thus $\lambda_1 \le \Delta(G)$.

The lower bound follows from the following fact:

Eigenvalues of $K_{1,\Delta}$ are $\sqrt{\Delta}, 0, \cdots, 0, -\sqrt{\Delta}$.

Idea 2. Eigenvalues interlace.

Déjà vu: Cauchy's Interlace Theorem

Let A be a symmetric matrix of size n, and B is a principal submatrix of A of size $m \le n$. Suppose the eigenvalues of A are

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n,$$

and the eigenvalues of B are

$$\mu_1 \ge \cdots \ge \mu_m.$$

Then for $1 \le i \le m$, we have

$$\lambda_{i+n-m} \le \mu_i \le \lambda_i.$$

Eigenvalue interlacing (II)

The eigenvalues of Q^n are:

$$n^{\binom{n}{0}}, (n-2)^{\binom{n}{1}}, \cdots, (n-2i)^{\binom{n}{i}}, \cdots, -n^{\binom{n}{n}}.$$

We have

$$\lambda_1(H) \geq \lambda_{2^{n-1}}(Q^n) \in \{0, 1\}.$$

It looks too trivial, yet from interlacing we have

A weaker proposition

If H is an induced subgraph of Q^n on $(\frac{1}{2} + c) \cdot 2^n$ vertices for some $c > 0$, then

$$\Delta(H) \geq c' \sqrt{n}.$$

Signed adjacency matrix (I)

Idea 3. Use a signed adjacency matrix.

Lemma

For every graph G, and M is a symmetric signed adjacency matrix of G with largest eigenvalue λ_1,

$$\lambda_1 \leq \Delta(G).$$

Proof is same as before.

If we can find such M, whose 2^{n-1}-th largest eigenvalue is $\sqrt{n}$, then we are done by interlacing!

Signed adjacency matrix (II)

Such matrix exists:

$$M_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_{i+1} = \begin{bmatrix} M_i & I \\ I & -M_i \end{bmatrix}.$$

Then by induction, $M_n^2 = nI$. Therefore the eigenvalues of M_n are

$$\{\sqrt{n}^{(2^{n-1})}, -\sqrt{n}^{(2^{n-1})}\}.$$

Hadamard's inequality

For a $m \times m$ matrix M with row vectors v_i,

$$|\det(M)| \le \prod_{i=1}^{m} \|v_i\|.$$

Equality is achieved if and only if all the row vectors are orthogonal.

- M is a signed adj. matrix of Q^n. $\Longrightarrow |\det(M)| \le (\sqrt{n})^{2^n}$.
- $\lambda_{2^{n-1}}(M)$ is at least $\sqrt{n}$. $\Longrightarrow \det(M) \ge (\sqrt{n})^{2^n}$.

Therefore, we need: $M^t M = nI$.

Future project

Chvátal's Conjecture (1974)

Given an abstract simplicial complex $\mathcal{F}$ (family of subsets such that $A \in \mathcal{F}$, $B \subset A$ implies $B \in \mathcal{F}$), the maximum size of its intersecting subfamily is always attained by sets of $\mathcal{F}$ containing a fixed element.

The conjecture is still wide open, except for some special cases including $\binom{[n]}{\le k} = \{\text{subsets of } [n] \text{ of size up to } k\}$.

123 124 134 234

12 13 14 23 24 34

1 2 3 4

$\varnothing$

Last but not least . . .

Thank you!