Limit Laws for q-Hook Formulas

Sara Billey
University of Washington

Based on joint work with:
Joshua Swanson

arXiv:2010.12701
Slides: `math.washington.edu/~billey/talks/hooks.pdf`

Triangle Lectures in Combinatorics
November 14, 2020

Outline

Motivating Example: q-enumeration of SYT's via major index

Generalized q-hook length formulas

Moduli space of limiting distributions for SSYTs and forests

Open Problems

Standard Young Tableaux

Defn. A *standard Young tableau* of shape λ is a bijective filling of λ such that every row is increasing from left to right and every column is increasing from top to bottom.

1	3	6	7	9
2	5	8		
4				

Important Fact. The standard Young tableaux of shape λ, denoted $\mathrm{SYT}(\lambda)$, index a basis of the irreducible S_n representation indexed by λ.

Counting Standard Young Tableaux

Hook Length Formula.(Frame-Robinson-Thrall, 1954)
If λ is a partition of n, then

$$\#SYT(\lambda) = \frac{n!}{\prod_{c \in \lambda} h_c}$$

where h_c is the *hook length* of the cell c, i.e. the number of cells directly to the right of c or below c, including c.

Example. Filling cells of $\lambda = (5, 3, 1) \vdash 9$ by hook lengths:

7	5	4	2	1
4	2	1		
1				

So, $\#SYT(5, 3, 1) = \frac{9!}{7 \cdot 5 \cdot 4 \cdot 2 \cdot 4 \cdot 2} = 162$.

Remark. Notable other proofs by Greene-Nijenhuis-Wilf '79 (probabilistic), Eriksson '93 (bijective), Krattenthaler '95 (bijective), Novelli -Pak -Stoyanovskii'97 (bijective), Bandlow'08,

q-Counting Standard Young Tableaux

Def. The *descent set* of a standard Young tableau T, denoted $D(T)$, is the set of positive integers i such that $i+1$ lies in a row strictly below the cell containing i in T.

The *major index* of T is the sum of its descents:

$$\mathrm{maj}(T) = \sum_{i \in D(T)} i.$$

Example. The descent set of T is $D(T) = \{1, 3, 4, 7\}$ so $\mathrm{maj}(T) = 15$ for $T =$

1	3	6	7	9
2	4	8		
5				

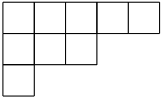
.

Def. The *major index generating function* for λ is

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)}$$

q-Counting Standard Young Tableaux

Example. $\lambda = (5, 3, 1)$



$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} =$

$$q^{23} + 2q^{22} + 4q^{21} + 5q^{20} + 8q^{19} + 10q^{18} + 13q^{17} + 14q^{16} + 16q^{15}$$

$$+16q^{14} + 16q^{13} + 14q^{12} + 13q^{11} + 10q^{10} + 8q^{9} + 5q^{8} + 4q^{7} + 2q^{6} + q^{5}$$

Note, at $q = 1$, we get back 162.

“Fast” Computation of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$

Thm.(Stanley's q-analog of the Hook Length Formula for $\lambda \vdash n$)

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \frac{q^{b(\lambda)}[n]_q!}{\prod_{c\in\lambda}[h_c]_q}$$

where

- $b(\lambda) := \sum(i-1)\lambda_i$
- h_c is the hook length of the cell c
- $[n]_q := 1 + q + \cdots + q^{n-1} = \frac{q^n-1}{q-1}$
- $[n]_q! := [n]_q[n-1]_q\cdots[1]_q$

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The Trick. Each q-integer $[n]_q$ factors into a product of *cyclotomic polynomials* $\Phi_d(q)$,

$$[n]_q = 1 + q + \cdots + q^{n-1} = \prod_{d|n}\Phi_d(q).$$

Cancel all of the factors from the denominator of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ from the numerator, and then expand the remaining product.

Corollaries of Stanley's formula

Thm.(Stanley's q-analog of the Hook Length Formula for $\lambda \vdash n$)

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \frac{q^{b(\lambda)}[n]_q!}{\prod_{c\in\lambda}[h_c]_q}$$

Corollaries.

1. $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = q^{b(\lambda)-b(\lambda')}\,\mathrm{SYT}(\lambda')^{\mathrm{maj}}(q)$.
2. The coefficients of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ are symmetric.
3. There is a unique min-maj and max-maj tableau of shape λ.

Motivation for q-Counting Standard Young Tableaux

Thm.(Lusztig-Stanley 1979) Given a partition $\lambda \vdash n$, say

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \sum_{k \geq 0} b_{\lambda,k} q^k.$$

Then $b_{\lambda,k} := \#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) = k\}$ is the number of times the irreducible S_n module indexed by λ appears in the decomposition of the coinvariant algebra $\mathbb{Z}[x_1, x_2, \ldots, x_n]/I_+$ in the homogeneous component of degree k.

Key Questions for $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$

Recall $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \sum b_{\lambda,k} q^k$.

Distribution Question. What patterns do the coefficients in the list $(b_{\lambda,0}, b_{\lambda,1}, \ldots)$ exhibit?

Existence Question. For which λ, k does $b_{\lambda,k} = 0$?

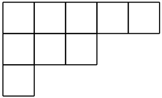
Unimodality Question. For which λ, are the coefficients of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ *unimodal*, meaning

$$b_{\lambda,0} \leq b_{\lambda,1} \leq \ldots \leq b_{\lambda,m} \geq b_{\lambda,m+1} \geq \ldots?$$

References: arXiv:1905.00975, arXiv:1809.07386.

q-Counting Standard Young Tableaux

Example. $\lambda = (5,3,1)$



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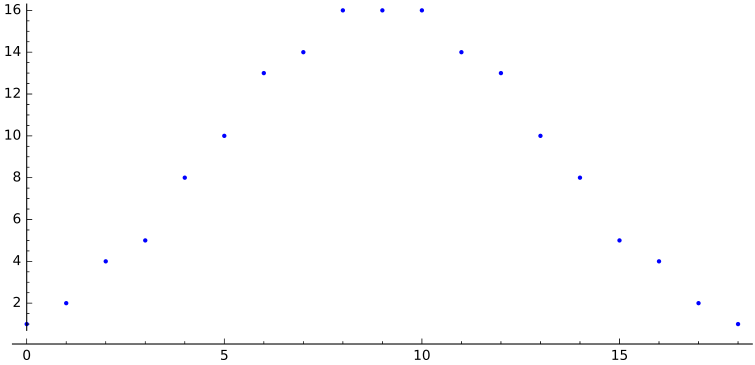
$$+16q^{14} + 16q^{13} + 14q^{12} + 13q^{11} + 10q^{10} + 8q^{9} + 5q^{8} + 4q^{7} + 2q^{6} + q^{5}$$

Notation: (00000 1 2 4 5 8 10 13 14 16 16 16 14 13 10 8 5 4 2 1)

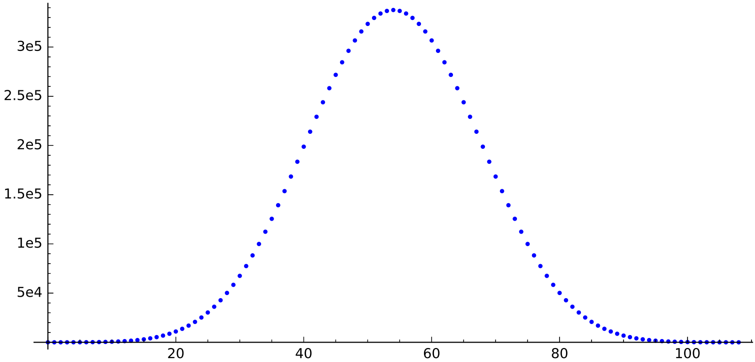
Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(5,3,1)^{\mathrm{maj}}(q)$:

$$(1,2,4,5,8,10,13,14,16,16,16,14,13,10,8,5,4,2,1)$$

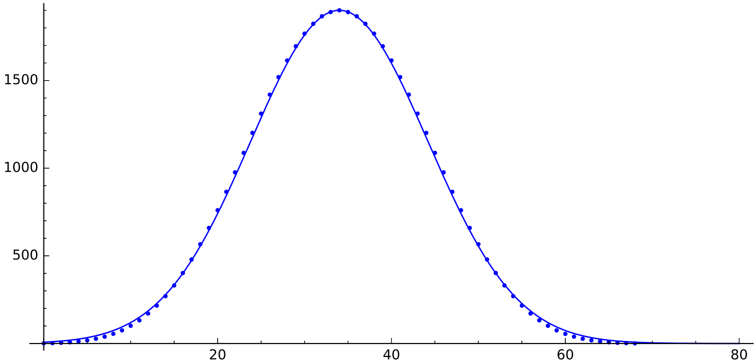
Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(11, 5, 3, 1)^{\mathrm{maj}}(q)$.

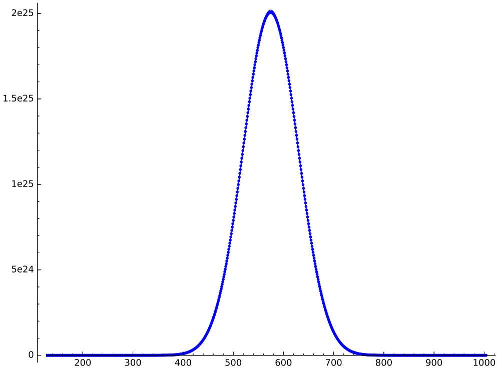
Question. What type of curve is that?

Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(10,6,1)^{\mathrm{maj}}(q)$ along with the Normal distribution with $\mu = 34$ and $\sigma^2 = 98$.

Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(8,8,7,6,5,5,5,2,2)^{\mathrm{maj}}(q)$ along with the corresponding normal distribution.

Converting q-Enumeration to Discrete Probability

Distribution Question. What is the limiting distribution(s) for the coefficients in $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$?

From Combinatorics to Probability.
If $f(q) = a_0 + a_1 q + a_2 q^2 + \cdots + a_n q^n$ where a_i are nonnegative integers, then construct the random variable X_f with discrete probability distribution

$$\mathbb{P}(X_f = k) = \frac{a_k}{\sum_j a_j} = \frac{a_k}{f(1)}.$$

If f is part of a family of q-analogs of an integer sequence, we can study the limiting distributions.

Converting q-Enumeration to Discrete Probability

Example. For $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum b_{\lambda,k} q^k$, define the integer random variable $X_\lambda[\mathrm{maj}]$ with discrete probability distribution

$$\mathbb{P}(X_\lambda[\mathrm{maj}] = k) = \frac{b_{\lambda,k}}{|\mathrm{SYT}(\lambda)|}.$$

We claim the distribution of $X_\lambda[\mathrm{maj}]$ "usually" is approximately normal for most shapes λ. Let's make that precise!

Standardization

Def. The *standardization* of $X_\lambda[\text{maj}]$ is

$$X_\lambda^*[\text{maj}] = \frac{X_\lambda[\text{maj}] - \mu_\lambda}{\sigma_\lambda}.$$

So $X_\lambda^*[\text{maj}]$ has mean 0 and variance 1 for any λ.

Thm.(Adin-Roichman, 2001)
For any partition λ, the mean and variance of $X_\lambda[\text{maj}]$ are

$$\mu_\lambda = \frac{\binom{|\lambda|}{2} - b(\lambda') + b(\lambda)}{2} = b(\lambda) + \frac{1}{2}\left[\sum_{j=1}^{|\lambda|} j - \sum_{c\in\lambda} h_c\right],$$

and

$$\sigma_\lambda^2 = \frac{1}{12}\left[\sum_{j=1}^{|\lambda|} j^2 - \sum_{c\in\lambda} h_c^2\right].$$

Asymptotic Normality

Def. Let $X_1, X_2, \ldots$ be a sequence of real-valued random variables with standardized cumulative distribution functions $F_1(t), F_2(t), \ldots$. The sequence is *asymptotically normal* if

$$\forall t \in \mathbb{R}, \qquad \lim_{n \to \infty} F_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} = \mathbb{P}(N < t)$$

where N is a Normal random variable with mean 0 and variance 1.

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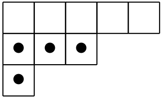
Question. In what way can a sequence of partitions approach infinity?

The Aft Statistic

Def. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, let

$$\text{aft}(\lambda) := n - \max\{\lambda_1, k\}.$$

Example. $\lambda = (5, 3, 1)$ then $\text{aft}(\lambda) = 4$.



Look it up: Aft is now on FindStat as St001214

Distribution Question: From Combinatorics to Probability

Thm.(Billey-Konvalinka-Swanson, 2019)

Suppose $\lambda^{(1)}, \lambda^{(2)}, \ldots$ is a sequence of partitions, and let $X_N := X_{\lambda^{(N)}}[\text{maj}]$ be the corresponding random variables for the maj statistic. Then, the sequence $X_1, X_2, \ldots$ is asymptotically normal if and only if $\text{aft}(\lambda^{(N)}) \to \infty$ as $N \to \infty$.

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Question. What happens if $\text{aft}(\lambda^{(N)})$ does not go to infinity as $N \to \infty$?

Distribution Question: From Combinatorics to Probability

Thm.(Billey-Konvalinka-Swanson, 2019)
Let $\lambda^{(1)}, \lambda^{(2)}, \ldots$ be a sequence of partitions. Then $(X_{\lambda^{(N)}}[\mathrm{maj}]^*)$ converges in distribution if and only if

(i) $\mathrm{aft}(\lambda^{(N)}) \to \infty$; or

(ii) $|\lambda^{(N)}| \to \infty$ and $\mathrm{aft}(\lambda^{(N)})$ is eventually constant; or

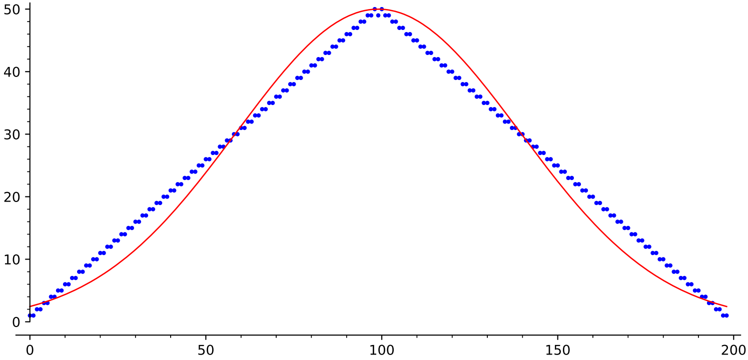
(iii) the distribution of $X^*_{\lambda^{(N)}}[\mathrm{maj}]$ is eventually constant.

The limit law is $\mathcal{N}(0,1)$ in case (i), $\mathcal{IH}^*_M$ in case (ii), and discrete in case (iii).

Here $\mathcal{IH}_M$ denotes the sum of M independent identically distributed uniform $[0,1]$ random variables, known as the Irwin–Hall distribution or the *uniform sum distribution*.

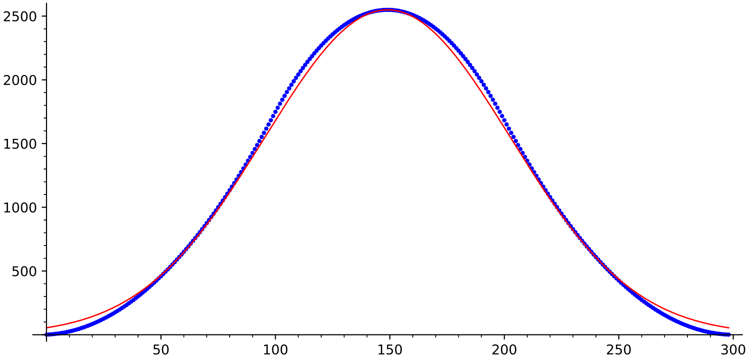
Distribution Question: From Combinatorics to Probability

Example. $\lambda = (100, 2)$ looks like the distribution of the sum of two independent uniform random variables on $[0, 1]$:



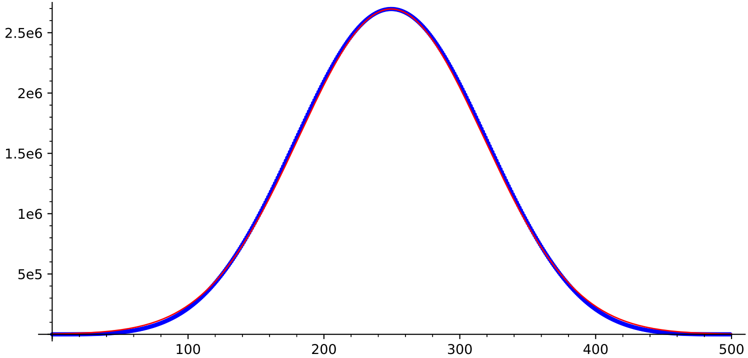
Distribution Question: From Combinatorics to Probability

Example. $\lambda = (100, 2, 1)$ looks like the distribution of the sum of three independent uniform random variables on $[0, 1]$:



Distribution Question: From Combinatorics to Probability

Example. $\lambda = (100, 3, 2)$ looks like the normal distribution, but not quite!



Proof ideas: Characterize the Moments and Cumulants

Definitions.

- For $d \in \mathbb{Z}_{\geq 0}$, the *dth moment*

$$\mu_d := \mathbb{E}[X^d]$$

- The *moment-generating function* of X is

$$M_X(t) := \mathbb{E}[e^{tX}] = \sum_{d=0}^{\infty} \mu_d \frac{t^d}{d!},$$

- The *cumulants* $\kappa_1, \kappa_2, \ldots$ of X are defined to be the coefficients of the exponential generating function

$$K_X(t) := \sum_{d=1}^{\infty} \kappa_d \frac{t^d}{d!} := \log M_X(t) = \log \mathbb{E}[e^{tX}].$$

Nice Properties of Cumulants

1. *(Familiar Values)* The first two cumulants are $\kappa_1 = \mu$, and $\kappa_2 = \sigma^2$.
2. *(Additivity)* The cumulants of the sum of *independent* random variables are the sums of the cumulants.
3. *(Homogeneity)* The dth cumulant of cX is $c^d \kappa_d$ for $c \in \mathbb{R}$.
4. *(Shift Invariance)* The second and higher cumulants of X agree with those for $X - c$ for any $c \in \mathbb{R}$.
5. *(Polynomial Equivalence)* The cumulants and moments are determined by polynomials in the other sequence.

Examples of Cumulants and Moments

Example. Let $X = \mathcal{N}(\mu, \sigma^2)$ be the normal random variable with mean μ and variance σ^2. Then the cumulants are

$$\kappa_d = \begin{cases} \mu & d = 1, \\ \sigma^2 & d = 2, \\ 0 & d \geq 3. \end{cases}$$

and for $d > 1$,

$$\mu_d = \begin{cases} 0 & \text{if } d \text{ is odd}, \\ \sigma^d (d-1)!! & \text{if } d \text{ is even}. \end{cases}$$

.

Example. For a Poisson random variable X with mean μ, the cumulants are all $\kappa_d = \mu$, while the moments are $\mu_d = \sum_{i=1}^{d} \mu_i S_{i,d}$.

Cumulants for Major Index Generating Functions

Thm.(Billey-Konvalinka-Swanson, 2019)
Let $\lambda \vdash n$ and $d \in \mathbb{Z}_{>1}$. If κ_d^λ is the dth cumulant of $X_\lambda[\mathrm{maj}]$, then

$$\kappa_d^\lambda = \frac{B_d}{d}\left[\sum_{j=1}^{n} j^d - \sum_{c\in\lambda} h_c^d\right] \tag{1}$$

where $B_0, B_1, B_2, \ldots = 1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \ldots$ are the Bernoulli numbers (OEIS A164555 / OEIS A027642).

Remark. We use this theorem to prove that as aft approaches infinity the standardized cumulants for $d \geq 3$ all go to 0 proving the Asymptotic Normality Theorem.

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Remark. We use this theorem to prove that as aft approaches infinity the standardized cumulants for $d \geq 3$ all go to 0 proving the Asymptotic Normality Theorem.

Remark. Note, κ_2^λ is exactly the Adin-Roichman variance formula.

Cumulants of certain q-analogs

Thm.(Chen–Wang–Wang-2008 and Hwang–Zacharovas-2015)
Suppose $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ are multisets of positive integers such that

$$f(q) = \frac{\prod_{j=1}^m [a_j]_q}{\prod_{j=1}^m [b_j]_q} = \sum c_k q^k \in \mathbb{Z}_{\geq 0}[q]$$

.

Let X be a discrete random variable with $\mathbb{P}(X = k) = c_k / f(1)$. Then the dth cumulant of X is

$$\kappa_d = \frac{B_d}{d} \sum_{j=1}^m (a_j^d - b_j^d)$$

where B_d is the dth Bernoulli number (with $B_1 = \frac{1}{2}$).

Example. This theorem applies to

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \frac{q^{b(\lambda)} [n]_q!}{\prod_{c \in \lambda} [h_c]_q}$$

Cyclotomic Generating Functions

Def. A polynomial $f(q)$ with nonnegative integer coefficients is a *cyclotomic generating function* provided it satisfies one of the following equivalent conditions:

(i) (Rational form.) There are multisets $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ of positive integers and $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ such that

$$f(q) = \alpha q^{\beta} \cdot \prod_{j=1}^{m} \frac{[a_j]_q}{[b_j]_q} = \alpha q^{\beta} \cdot \prod_{j=1}^{m} \frac{1 - q^{a_j}}{1 - q^{b_j}}. \tag{2}$$

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(ii) (Cyclotomic form.) The polynomial $f(q)$ can be written as a non-negative integer times a product of cyclotomic polynomials and factors of q.

(iii) (Complex form.) The complex roots of $f(q)$ are each either a root of unity or zero.

Cyclotomic Generating Functions

More examples of cyclotomic generating functions, aka q-hook length type formulas..

1. Stanley: $s_\lambda(1, q, q^2, \ldots, q^m)$.
2. Björner-Wachs: q-hook length formula for forests.
3. Macaulay: Hilbert series of polynomial quotients $k[x_1, \ldots, x_n]/(\theta_1, \theta_2, \ldots, \theta_n)$ where $deg(x_i) = b_i$, $deg(\theta_i) = a_i$, and $(\theta_1, \theta_2, \ldots, \theta_n)$ is a homogeneous system of parameters.
4. Chevalley: Length generating function restricted to minimum length coset representatives of a finite reflection group modulo a parabolic subgroup.
5. Iwahori-Matsumoto, Stembridge-Waugh, Zabrocki: Coxeter length generating function restricted to coset representatives of the extended affine Weyl group of type A_{n-1} mod translations by coroots. The associated statistic is $\mathrm{baj} - \mathrm{inv}$.

Cyclotomic Generating Functions

Remark. Corresponding with each cyclotomic generating function $f(q)$, there is a discrete random variable X_f supported on $\mathbb{Z}_{\geq 0}$ with probability generating function $f(q)/f(1)$ and higher cumulants for $d \geq 2$,

$$\kappa_d^f = \frac{B_d}{d} \sum_{j=1}^{m} (a_j^d - b_j^d).$$

Therefore, we can study asymptotics for interesting sequences of cyclotomic generating functions much like SYT.

Recent Progress based on joint work with Josh Swanson

1. MacMahon: q-counting plane partitions in box.
2. Stanley-Littlewood: $s_\lambda(1, q, q^2, \ldots, q^m)$.
3. Björner-Wachs: q-hook length formula for forests

MacMahon: q-counting plane partitions in box.

Let $\mathrm{PP}(a \times b \times c)$ be the set of all *plane partitions* that fit inside an $a \times b \times c$ box. Plane partitions can be represented by tableaux with decreasing rows and columns. The *size* of a plane partition is the sum of the numbers in the tableau.

MacMahon's Formula.

$$\sum_{T \in \mathrm{PP}(a \times b \times c)} q^{|T|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{[i+j+k-1]_q}{[i+j+k-2]_q}.$$

MacMahon: q-counting plane partitions in box.

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MacMahon's Formula is a cyclotomic generating function. Let $\mathcal{X}_{a\times b\times c}[\mathrm{size}]^*$ the corresponding random variable.

Recent Progress based on joint work with Josh Swanson

Recall, $\mathcal{N}(0,1)$ is the standard normal distribution, and $\mathcal{IH}_M = \sum_{i=1}^{M} \mathcal{U}[0,1]$ is the Irwin-Hall distribution.

Theorem. Let a, b, c each be a sequence of positive integers.

(i) $\mathcal{X}_{a\times b\times c}[\text{size}]^* \Rightarrow \mathcal{N}(0,1)$ if and only if median$\{a, b, c\} \to \infty$.

(ii) $\mathcal{X}_{a\times b\times c}[\text{size}]^* \Rightarrow \mathcal{IH}_M$ if $ab \to M < \infty$ and $c \to \infty$.

The limit of the median value determines the limiting distribution for plane partitions, just like aft determined the limiting distribution for *SYTs*.

Moduli space of standardized distributions

Motivating Philosophy. By the Central Limit Theorem, $\lim_{M\to\infty} \mathcal{IH}_M^* \Rightarrow \mathcal{N}(0,1)$, so instead of parametrizing the Irwin-Hall distributions by $\{n \in \mathbb{Z}_{\geq 1}\}$, use the parameter space

$$\mathbf{P}_{\mathcal{IH}} := \left\{ \frac{1}{n} : n \in \mathbb{Z}_{\geq 1} \right\} \subset \mathbb{R}$$

to get a related topological structure.

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Def. The *moduli space of Irwin-Hall distributions* is

$$\mathbf{M}_{\mathcal{IH}} := \{\mathcal{IH}^*_M : M \in \mathbb{Z}_{\geq 0}\},$$

Endow $\mathbf{M}_{\mathcal{IH}}$ with the topology characterized by convergence in distribution using the Lévy metric.

Moduli space of standardized distributions

Conclusions.

1. $\overline{\mathbf{P}_{\mathcal{IH}}} = \mathbf{P}_{\mathcal{IH}} \sqcup \{0\}$.

2. $\overline{\mathbf{M}_{\mathcal{IH}}} = \mathbf{M}_{\mathcal{IH}} \cup \{\mathcal{N}(0,1)\}$.

3. The bijection $\overline{\mathbf{P}_{\mathcal{IH}}} \to \overline{\mathbf{M}_{\mathcal{IH}}}$ given by $\frac{1}{M+1} \mapsto \mathcal{IH}^*_M$ and $0 \mapsto \mathcal{N}(0,1)$ is a homeomorphism.

Moduli space of plane partition distributions

Def. The *moduli space of plane partition distributions* is

$$\mathbf{M}_{\mathrm{PP}} := \{\mathcal{X}_{a\times b\times c}[\mathrm{size}]^* : a, b, c \in \mathbb{Z}_{\geq 1}\}$$

with the topology characterized by convergence in distribution.

Corollary. In the Lévy metric,

$$\overline{\mathbf{M}_{\mathrm{PP}}} = \mathbf{M}_{\mathrm{PP}} \sqcup \overline{\mathbf{M}_{\mathcal{IH}}},$$

which is compact. The set of limit points of $\mathbf{M}_{\mathrm{PP}}$ is exactly $\overline{\mathbf{M}_{\mathcal{IH}}}$.

Moduli space of SYT distributions

Def. The *moduli space of* SYT *distributions* is

$$\mathbf{M}_{\mathsf{SYT}} := \{X_\lambda[\mathrm{maj}]^* : \lambda \in \mathrm{Par}, \#SYT(\lambda) > 1\}$$

with the topology characterized by convergence in distribution.

Corollary. In the Lévy metric,

$$\overline{\mathbf{M}_{\mathsf{SYT}}} = \mathbf{M}_{\mathsf{SYT}} \sqcup \overline{\mathbf{M}_{\mathcal{IH}}},$$

which is compact. The set of limit points of $\mathbf{M}_{\mathsf{SYT}}$ is exactly $\overline{\mathbf{M}_{\mathcal{IH}}}$.

Semistandard tableaux and Schur functions

Defn. A *semistandard Young tableau* of shape λ is filling of λ such that every row is weakly increasing from left to right and every column is strictly increasing from top to bottom.

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 3 & 3 & 3 \\ \hline 2 & 5 & 5 \\ \cline{1-3} 9 \\ \cline{1-1}\end{array} \qquad x^T = x_1 x_2 x_3^4 x_5^2 x_9 \qquad \text{rank}(T) = 28$$

Associate a monomial to each semistandard tableau, $T \mapsto x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ where α_i is the number of i's in T. Let $\text{rank}(T) = \sum (i-1)\alpha_i$.

Def. The *Schur polynomial* indexed by λ on $(x_1, \ldots, x_m)$ is

$$s_\lambda(x_1, x_2, \ldots, x_m) = \sum x^T$$

summed over all semistandard Young tableaux of shape λ filled with numbers in $\{1, 2, \ldots, m\}$, denoted $\text{SSYT}_{\leq m}(\lambda)$.

Semistandard tableaux and Schur functions

Stanley+Littlewood. The principle specialization of the Schur polynomial is a cyclotomic generating function

$$\begin{aligned} s_\lambda(1, q, q^2, \ldots, q^{m-1}) &= \sum_{T \in \mathrm{SSYT}_{\leq m}(\lambda)} q^{\mathrm{rank}(T)} \\ &= q^{b(\lambda)} \prod_{u \in \lambda} \frac{[m + c_u]_q}{[h_u]_q} \\ &= q^{b(\lambda)} \prod_{1 \leq i < j \leq m} \frac{[\lambda_i - \lambda_j + j - i]_q}{[j - i]_q} \end{aligned}$$

where $c_u = j - i$ is the *content* of cell $u = (i, j)$ and h_u is the hook length of u.

Moduli Space of SSYT Distributions

Def. Let $\mathcal{X}_{\lambda;m}[\mathrm{rank}]$ denote the random variable associated with the rank statistic on $\mathrm{SSYT}_{\leq m}(\lambda)$, sampled uniformly at random.

Def. The *moduli space of* SSYT *distributions* is

$$\mathbf{M}_{\mathrm{SSYT}} := \{\mathcal{X}_{\lambda;m}[\mathrm{rank}]^* : \lambda \in \mathrm{Par}, \ell(\lambda) \leq m\}$$

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Open Problem. Describe $\overline{\mathbf{M}_{\mathrm{SSYT}}}$ in the Lévy metric. What are all possible limit points?

Toward Limit Laws of SSYT Distributions

Def. Given a finite multiset $\mathbf{t} = \{t_1 \geq t_2 \geq \cdots \geq t_m\}$ of non-negative real numbers, let

$$\mathcal{S}_{\mathbf{t}} := \sum_{t \in \mathbf{t}} \mathcal{U}\left[-\frac{t}{2}, \frac{t}{2}\right], \tag{3}$$

where we assume the summands are independent and $\mathcal{U}[a, b]$ denotes the continuous uniform distribution supported on $[a, b]$. We say $\mathcal{S}_{\mathbf{t}}$ is a *finite generalized uniform sum distribution*.

Example. If $\mathbf{t}$ consists of M copies of 1, then $\mathcal{S}_{\mathbf{t}} + \frac{M}{2}$ is the Irwin-Hall distribution $\mathcal{IH}_M$.

Distance Multisets

Def. The *distance multiset* of $\mathbf{t} = \{t_1 \geq t_2 \geq \cdots \geq t_m\}$ is the multiset

$$\Delta\mathbf{t} := \{t_i - t_j : 1 \leq i < j \leq m\}.$$

Theorem. Let λ be an infinite sequence of partitions with $\ell(\lambda) < m$ where $\lambda_1/m^3 \to \infty$. Let $\mathbf{t}(\lambda) = (t_1, \ldots, t_m) \in [0,1]^m$ be the finite multiset with $t_k := \frac{\lambda_k}{\lambda_1}$ for $1 \leq k \leq m$. Then $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^*$ converges in distribution if and only if the multisets $\Delta\mathbf{t}(\lambda)$ converge pointwise.

In that case, the limit distribution is $\mathcal{N}(0,1)$ if $m \to \infty$ and $\mathcal{S}^*_{\mathbf{d}}$ where $\Delta\mathbf{t}(\lambda) \to \mathbf{d}$ if m is bounded.

Moduli Space of Distance Distributions

Def. The *moduli space of distance distributions* is

$$\mathbf{M}_{\mathrm{DIST}} := \bigcup_{m\geq 2} \{\mathcal{S}^*_{\Delta\mathbf{t}} : \mathbf{t} = \{1 = t_1 \geq \cdots \geq t_m = 0\}\}$$

and its associated parameter space $\mathbf{P}_{\mathrm{DIST}}$ is a renormalized variation on $\Big\{\Delta\mathbf{t} : \mathbf{t} = \{1 = t_1 \geq \cdots \geq t_m = 0\}\Big\}$.

Conclusions/Thm.

1. $\overline{\mathbf{P}_{\mathrm{DIST}}} = \mathbf{P}_{\mathrm{DIST}} \sqcup \{\mathbf{0}\}$ where $\mathbf{0}$ is the infinite sequence of 0's.
2. $\overline{\mathbf{M}_{\mathrm{DIST}}} = \mathbf{M}_{\mathrm{DIST}} \sqcup \{\mathcal{N}(0,1)\}$.
3. The map $\overline{\mathbf{P}_{\mathrm{DIST}}} \to \overline{\mathbf{M}_{\mathrm{DIST}}}$ given by $\mathbf{d} \mapsto \mathcal{S}^*_{\mathbf{d}}$ and $\mathbf{0} \mapsto \mathcal{N}(0,1)$ is a homeomorphism between compact spaces.

Moduli Space of SSYT Distributions

Corollary. For any fixed $\epsilon > 0$, let

$$\mathbf{M}_{\epsilon\,\mathsf{SSYT}} := \{\mathcal{X}_{\lambda;m}[\mathrm{rank}]^* : \ell(\lambda) < m \text{ and } \lambda_1/m^3 > (|\lambda|+m)^\epsilon\} \subset \mathbf{M}_{\mathsf{SSYT}}.$$

Then

$$\overline{\mathbf{M}_{\epsilon\,\mathsf{SSYT}}} = \mathbf{M}_{\epsilon\,\mathsf{SSYT}} \sqcup \overline{\mathbf{M}_{\mathsf{DIST}}},$$

which is compact. The set of limit points of $\mathbf{M}_{\epsilon\,\mathsf{SSYT}}$ is $\overline{\mathbf{M}_{\mathsf{DIST}}}$.

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Then

$$\overline{\mathbf{M}_{\epsilon\,\mathrm{SSYT}}} = \mathbf{M}_{\epsilon\,\mathrm{SSYT}} \sqcup \overline{\mathbf{M}_{\mathrm{DIST}}},$$

which is compact. The set of limit points of $\mathbf{M}_{\epsilon\,\mathrm{SSYT}}$ is $\overline{\mathbf{M}_{\mathrm{DIST}}}$.

Corollary. For the moduli space of limit laws for Stanley's q-hook-content formula, we have shown

$$\mathbf{M}_{\mathrm{SSYT}} \cup \mathbf{M}_{\mathrm{DIST}} \cup \mathbf{M}_{\mathcal{IH}} \cup \{\mathcal{N}(0,1)\} \subset \overline{\mathbf{M}_{\mathrm{SSYT}}}.$$

Moduli Space of Generalized Sum Distributions

The limiting distributions q-hook length formulas for linear extensions of forests due to Björner–Wachs include all countably infinite generalized uniform sum distributions with finite variance, which is closely related to the 2-norm of the indexing multiset.

Theorem. The limit laws for all possible standardized general uniform sum distributions $\mathbf{M}_{\mathrm{SUMS}} : \{\mathcal{S}_{\mathbf{t}}^{*} : \mathbf{t} \in \widetilde{\ell_2}\}$ is exactly the *moduli space of DUSTPAN distributions,*

$$\overline{\mathbf{M}_{\mathrm{SUMS}}} = \mathbf{M}_{\mathrm{DUST}} := \{\mathcal{S}_{\mathbf{t}} + \mathcal{N}(0, \sigma^2) : |\mathbf{t}|_2^2/12 + \sigma^2 = 1\}.$$

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The nomenclature DUSTPAN refers to a *distribution associated to a uniform sum for* $\mathbf{t}$ *plus an independent normal distribution.*

The moduli space of limit laws for q-hook formulas

Let $\mathbf{M}_{\text{Forest}}$ be the moduli space of standardized distributions associated to forests. We know $\mathbf{M}_{\text{Forest}} \cup \mathbf{M}_{\text{DUST}} \subset \overline{\mathbf{M}_{\text{Forest}}}$, implying there are an uncountable number of possible limit laws for distributions associated to forests.

Open Problem. Describe $\overline{\mathbf{M}_{\text{Forest}}}$ in the Lévy metric. What are all possible limit points?

Open Problem. Describe $\overline{\mathbf{M}_{\text{CGF}}}$ in the Lévy metric. What are all possible limit points? Is $\mathbf{M}_{\text{CGF}} \cup \mathbf{M}_{\text{DUST}}$ <u>the</u> moduli space of limit laws for q-hook formulas?

Conclusion

Many Thanks!