MA 721 – Midterm II

Spring 2020

1. (10 points) Here is a matrix in rational canonical form:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

(a) What are its invariant factors?

(b) What is its Jordan canonical form?

(c) What is its minimal polynomial?

(d) What is its characteristic polynomial?

Solution:

(a) $a_1 = x^3 - x^2$, $a_2 = x^4 - 2x^3 + x^2$

(b) Jordan canonical form: $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$

(c) minimal polynomial $= x^2(x-1)^2 = x^4 - 2x^3 + x^2$

(d) characteristic polynomial $= x^4(x-1)^3 = x^7 - 3x^6 + 3x^5 - x^4$

2. (10 points) Consider the $\mathbb{Q}$-vectorspace

$$V = \mathbb{Q}[x]/\langle (x-\lambda)^{d_1}\rangle \oplus \mathbb{Q}[x]/\langle (x-\lambda)^{d_2}\rangle \oplus \ldots \oplus \mathbb{Q}[x]/\langle (x-\lambda)^{d_k}\rangle$$

where $0 < d_1 \leq d_2 \leq \ldots \leq d_k$ and $\lambda \in \mathbb{Q}$. Let $T : V \to V$ be the linear transformation given by multiplication by x. For each of the following, do not justify your answers:

(a) What is the minimal polynomial of T?

(b) What is the characteristic polynomial of T?

(c) What is the dimension of the kernel of $T - \lambda \cdot \text{id}$?
(Here id denotes the identity map $V \to V$.)

(d) Suppose $p(x) \in \mathbb{Q}[x]$ and $p(\lambda) \neq 0$. Let $L : V \to V$ be the linear map given by multiplication by $p(x)$. What is the dimension of the kernel of L?

Solution:

(a) minimal polynomial of $T = (x-\lambda)^{d_k}$

(b) characteristic polynomial of $T = (x-\lambda)^n$ where $n = \sum_{i=1}^{k} d_i$

(c) $\dim(\ker(T - \lambda \cdot \text{id})) = k$

(d) $\dim(\ker(L)) = 0$

3. (15 points) For each pair of principal ideal domain R and R-module M, give the **rank**, **invariant factors**, and **elementary divisors** of M. Do not justify your answers.

   (a) $R = \mathbb{Z}$, $M = \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Z}/35\mathbb{Z}$

   (b) $R = \mathbb{Z}$, $M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})$

   (c) $R = \mathbb{Z}$, $M = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$

   (d) $R = \mathbb{Z}/5\mathbb{Z}$, $M = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$

   (e) $R = \mathbb{Z}$, $M = \mathbb{Z}^2/N$ where $N \subseteq \mathbb{Z}^2$ is the submodule generated by $\{(2, 1), (1, -1)\}$

Solution:

	rank	invariant factors	elementary divisors
(a)	0	$5,\ 5,\ 2 \cdot 3 \cdot 5 \cdot 7 = 210$	2, 3, 5, 5, 5, 7
(b)	2	3, 3	3, 3
(c)	0	2	2
(d)	2	$\emptyset$	$\emptyset$
(e)	0	3	3

4. (10 points) Here is a primary decomposition of an ideal in $\mathbb{Q}[x, y, z]$:

$$I = \langle x^2z, xy^2, xyz, xz^2 \rangle = \langle x \rangle \cap \langle y^2, z \rangle \cap \langle x, y, z \rangle^3.$$

(a) List the associated primes of I and label each as isolated or embedded.

(b) For each associated prime P, give an element $f \in R$ for which $P = \text{rad}(I : \langle f \rangle)$.

Solution:

(a) We see that that the decomposition above is minimal:

$$\begin{aligned} xz &\in \big(\langle x \rangle \cap \langle y^2, z \rangle\big) \backslash \langle x, y, z \rangle^3 \\ x^3 &\in \big(\langle x \rangle \cap \langle x, y, z \rangle^3\big) \backslash \langle y^2, z \rangle \\ y^2z &\in \big(\langle y^2, z \rangle \cap \langle x, y, z \rangle^3\big) \backslash \langle x \rangle \end{aligned}$$

Moreover the prime ideals

$$\text{rad}(\langle x \rangle) = \langle x \rangle, \quad \text{rad}(\langle y^2, z \rangle) = \langle y, z \rangle, \quad \text{rad}(\langle x, y, z \rangle^3) = \langle x, y, z \rangle$$

are distinct. Note that $\langle x \rangle \subset \langle x, y, z \rangle$, but $\langle x \rangle$ and $\langle y, z \rangle$ do not contain either of the other ideals. Therefore we have

$$\begin{aligned} \text{Embedded primes: } &\langle x, y, z \rangle \\ \text{Isolated primes: } &\langle x \rangle, \ \langle y, z \rangle \end{aligned}$$

(b) For $P = \langle x \rangle$, consider $f = y^2z$. Since $y^2z \in \langle y^2, z \rangle \cap \langle x, y, z \rangle^3$, but $y^2z \notin \langle x \rangle$, $\text{rad}(I : \langle y^2z \rangle) = \langle x \rangle$.

Since $x^3 \in \langle x \rangle \cap \langle x, y, z \rangle^3$, but $x^3 \notin \langle y^2, z \rangle$, $\text{rad}(I : \langle x^3 \rangle) = \langle y^2, z \rangle$.

Since $xz \in \langle x \rangle \cap \langle y^2, z \rangle$, but $xz \notin \langle x, y, z \rangle^3$, $\text{rad}(I : \langle xz \rangle) = \langle x, y, z \rangle$

In summary,

$$\text{rad}(I : \langle y^2z \rangle) = \langle x \rangle, \quad \text{rad}(I : \langle x^3 \rangle) = \langle y, z \rangle, \quad \text{and} \quad \text{rad}(I : \langle xz \rangle) = \langle x, y, z \rangle$$

5. (10 points) Let R be a Noetherian ring and $Q \subseteq R$ an ideal.

   (a) Using the definition, show that Q is primary if and only if every zero-divisor of R/Q is nilpotent. (Recall that an element r is nilpotent if $r^k = 0$ for some k.)

   (b) For $R = \mathbb{Q}[x, y]$ and $Q = \langle x^2, xy\rangle$ give an example of a zero-divisor in R/Q that is *not* nilpotent.

Solution:

(a) ($\Rightarrow$) Suppose that Q is primary and let $r + Q$ be a zero divisor in R/Q. Then $r + Q$ is nonzero and there exists a nonzero element $s + Q \in R/Q$ so that $(r+Q)\cdot(s+Q) = 0+Q$. Then $rs \in Q$ and since $s+Q$ is nonzero, $s \notin Q$. Since Q is primary, this implies that $r^k \in Q$ for some k. Then $(r+Q)^k = r^k+Q = 0+Q$, meaning that $r + Q$ is a zero divisor in R/Q.

($\Leftarrow$) Suppose that every zero-divisor of R/Q is nilpotent. Suppose that for some $r, s \in R$, $rs \in Q$ and $s \notin Q$. Then $(r + Q)(s + Q) = rs + Q = 0 + Q$.

If $r \in Q$, then some power of r belongs to Q.

If $r \notin Q$, then $r+Q$ and $s+Q$ are non-zero in R/Q and therefore zero-divisors. By assumption, $r + Q$ is nilpotent, meaning that $(r + Q)^k = r^k + Q = 0 + Q$ for some k. It follows that $r^k \in Q$.

Since $rs \in Q$ and $s \notin Q$ implies $r^k \in Q$ for some k, Q is primary.

(b) Consider $y + Q$ in R/Q.

To see that this is a zero-divisor in R/Q, note that $x \notin Q$ and $y \notin Q$, so $x + Q$ and $y+Q$ are nonzero in R/Q. Moreover $xy \in Q$, so $(x+Q)(y+Q) = xy+Q = 0 + Q$, meaning that $y + Q$ is a zero-divisor.

It is not nilpotent because $(y + Q)^k = y^k + Q$. Note that every element of Q has the form $x^2 \cdot f + xy \cdot g$, which is divisible by x. In particular, $y^k \notin Q$ for any k. Therefore $(y + Q)^k \neq 0$ for any k.

6. (15 points) For each, answer **True** or **False**. Do not justify your answer.

   (a) A linear transformation $T : V \to V$ on a finite dimensional vectorspace V over an algebraically closed field is a direct sum of its generalized eigenspaces.

   (b) If two matrices have the same minimal and characteristic polynomials, then they are similar.

   (c) If R is a Noetherian ring and $I \subseteq R$ is an ideal, then R/I is also Noetherian.

   (d) If I is an ideal of a principal ideal domain R, then I has no embedded primes.

   (e) The intersection of prime ideals is prime.

Solution:

(a) True.

(b) False.

(c) True.

(d) True.

(e) False.