# MA 721 – Midterm I

Spring 2020

# Solutions

Name: ______________________________

| Question | Points | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 15 | |
| Total: | 70 | |

1. (a) (5 points) For each of the following, give a simpler description the module up to isomorphism (e.g. $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}/15\mathbb{Z}$). Do not justify your answers.

$$\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z}$$

$$\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/5\mathbb{Z} \cong \{0\}$$

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^2 \cong \mathbb{Q}^2$$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}) \cong \{0\}$$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{Z}/3\mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^2$$

(b) (10 points) Suppose that $V$ and $W$ are vectorspaces over a field $F$ of dimensions $n$ and $m$, respectively. For each of the following $F$-vectorspaces give the dimension and a basis, assuming that $\{v_1, \ldots, v_n\}$ is a basis for $V$ and $\{w_1, \ldots, w_m\}$ is a basis for $W$. Do not justify your answers.

| Module | Dimension | Basis |
|---|---|---|
| $V \oplus W$ | $m+n$ | $\{(v_i, 0), (0, w_j) : i = 1, \ldots, n, j = 1, \ldots, m\}$ |
| $V \otimes_F W$ | $m \cdot n$ | $\{v_i \otimes w_j : i = 1, \ldots, n, j = 1, \ldots, m\}$ |
| $\mathrm{Hom}_F(V, W)$ | $m \cdot n$ | $\{\varphi_{ij} : i = 1, \ldots, n, j = 1, \ldots, m\}$<br>where $\varphi_{ij}(v_i) = w_j$ and $\varphi_{ij}(v_k) = 0$ for $k \neq i$ |
| $\mathcal{S}^3(V)$ | $\binom{n+2}{3}$ | $\{v_{i_1} \otimes v_{i_2} \otimes v_{i_3} + \mathcal{C}^3(V) : 1 \leq i_1 \leq i_2 \leq i_3 \leq n\}$ |
| $\bigwedge(V)$ | $2^n$ | $\{v_{i_1} \wedge \cdots \wedge v_{i_k} : 0 \leq k \leq n,\ 1 \leq i_1 < \ldots < i_k \leq n\}$ |

2. Let $A$, $B$, and $C$ be $R$-modules.

   (a) (5 points) Give an explicit isomorphism

$$\Phi : (A \oplus B) \otimes_R C \to (A \otimes_R C) \oplus (B \otimes_R C)$$

   by giving values of $\Phi$ and its inverse on simple tensors.
   Do not justify your answers.

**Solution:**
$\Phi((a, b) \otimes c)) = (a \otimes c, b \otimes c)$

$\Phi^{-1}(a \otimes c_1, b \otimes c_2) = (a, 0) \otimes c_1 + (0, b) \otimes c_2$

   (b) (5 points) Similarly, describe an explicit isomorphism

$$\text{Hom}_R(A \oplus B, C) \to \text{Hom}_R(A, C) \oplus \text{Hom}_R(B, C)$$

   and its inverse. Do not justify your answers.

**Solution:** Define $\Phi : \text{Hom}_R(A \oplus B, C) \to \text{Hom}_R(A, C) \oplus \text{Hom}_R(B, C)$ as follows. For a homomorphism $\varphi \in \text{Hom}_R(A \oplus B, C)$, define

$$\Phi(\varphi) = (\psi_1, \psi_2) \text{ where } \psi_1(a) = \varphi(a, 0) \text{ and } \psi_2(b) = \varphi(0, b)$$

For $(\psi_1, \psi_2) \in \text{Hom}_R(A, C) \oplus \text{Hom}_R(B, C)$,

$$\Phi^{-1}(\psi_1, \psi_2) = \varphi \quad \text{where} \quad \varphi(a, b) = \psi_1(a) + \psi_2(b).$$

3. (10 points) Given an $R$-module $M$, the set of torsion elements of $M$ is defined to be

$$\mathrm{Tor}(M) = \{m \in M : rm = 0 \text{ for some } r \in R \backslash \{0\}\}.$$

Show that if $R$ is an integral domain, then $\mathrm{Tor}(M)$ is a submodule of $M$.

**Solution:** To check that $\mathrm{Tor}(M)$ is a submodule of $M$, we use the submodule criterion. First note that $0 \in \mathrm{Tor}(M)$ and so it is nonempty.

Next, let $r \in R$ and $x, y \in \mathrm{Tor}(M)$. Then there exist $s, t \in R\backslash\{0\}$ for which $sx = 0$ and $ty = 0$. Since $R$ is an integral domain $s \neq 0$ and $t \neq 0$ implies that $st \neq 0$. Also, $R$ is necessarily commutative. Then

$$st \cdot (x + ry) = stx + stry = t \cdot (sx) + rs \cdot (ty) = t \cdot (0) + rs \cdot (0) = 0.$$

Since $st \in R\backslash\{0\}$ it follows that $x + ry \in \mathrm{Tor}(M)$.

4. (10 points) Let $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ be an exact sequence of $R$-modules that splits. Prove that $B \cong A \oplus C$.

**Solution:** By definition, there is a homomorphism $\mu : C \to B$ for which $\psi \circ \mu = \mathrm{id}_C$. Then both $\varphi(A)$ and $\mu(C)$ are submodules of $B$. Since $\varphi(A) = \ker(\psi)$ and $\psi$ defines an injective map on $\mu(C)$, it must be that $\varphi(A) \cap \mu(C) = \{0\}$.

To see that $B = \varphi(A) + \mu(C)$, let $b \in B$. Then $c = \psi(b) \in C$. More over

$$\psi(b - \mu(c)) \;=\; \psi(b) - \psi(\mu(c)) \;=\; c - c \;=\; 0.$$

Therefore $b - \mu(c) \in \ker(\psi) = \varphi(A)$, meaning that there is some $a \in A$ with

$$\varphi(a) = b - \mu(c) \;\Rightarrow\; b = \varphi(a) + \mu(c).$$

This shows that $B \cong \varphi(A) \oplus \mu(C)$. Since $\varphi$ is injective on $A$ and $\mu$ is injective on $C$, we have that $A \cong \varphi(A)$ and $C \cong \mu(C)$. Therefore

$$B \cong \varphi(A) \oplus \mu(C) \cong A \oplus C.$$

5. Let $R$ denote the ring $\mathbb{Z} \oplus \mathbb{Z}$ under coordinate-wise addition and multiplication.

   (a) (5 points) Show that $M = \mathbb{Z}$ is an $R$-module with $(a, b) \cdot z = az$.

**Solution:**
Note that $M = \mathbb{Z}$ is an abelian group under usual addition.

Let $(a, b), (c, d) \in R = \mathbb{Z} \oplus \mathbb{Z}$ and $z, w \in M = \mathbb{Z}$. Then

- $((a,b)+(c,d)) \cdot z = (a+c, b+d) \cdot z = (a+c) \cdot z = az+cz = (a,b) \cdot z+(c,d) \cdot z$,
- $((a,b) \cdot (c,d)) \cdot z = (ac, bd) \cdot z = (ac) \cdot z = acz = (a,b) \cdot cz = (a,b) \cdot ((c,d) \cdot z)$,
- $(a,b) \cdot (z+w) = a(z+w) = az+aw = (a,b) \cdot z + (a,b) \cdot w$
- $(1,1)$ is the multiplicative identity in $R$ and $(1,1) \cdot z = 1z = z$.

Therefore this action makes $M = \mathbb{Z}$ into an $R$-module.

   (b) (5 points) Circle all that apply. Do not justify your answers.

As an $R$-module, $M = \mathbb{Z}$ is . . .

free [✓ projective] injective [✓ flat] [✓ finitely generated]

6. (15 points) True or false. Circle one. Do not justify your answers.

   (a) An $R$-module $M$ is finitely-generated if and only if it is a quotient of a free module $R^n$ for some $n \in \mathbb{Z}_{\geq 0}$.

   [✓ True] False

   (b) An $R$-module homomorphism $\Phi : M \otimes_R M \to N$ is injective if and only if the $R$-bilinear map $\phi : M \times M \to N$ given by $\phi(m_1, m_2) = \Phi(m_1 \otimes m_2)$ is injective.

   True [✓ False]

   (c) If $R$ is a commutative ring with $1_R \neq 0$, then every $R$-algebra is commutative.

   True [✓ False]

   (d) For every $R$-module homomorphism $\varphi : A \to B$ there are submodules $A' \subseteq A$ and $B' \subseteq B$ for which the sequence $0 \to A' \xrightarrow{i} A \xrightarrow{\varphi} B' \to 0$ is exact, where the map $i : A' \to A$ is given by inclusion.

   [✓ True] False

   (e) $\langle x^2 \rangle$ is a graded ideal in $\mathbb{Q}[x, y]$.

   [✓ True] False