MA 721 – Midterm I

Spring 2020

Time: 75 mins.

1. Answer all questions in the spaces provided. If you run out of room for an answer, continue on the back of the page. There is also two blank pages at the end for scratch work or continued answers.
2. Unless stated otherwise, justify your answers to receive full credit. Your answers do not have to be in complete sentences, but they do need to be understandable.
3. No calculators, notes, or other outside assistance is allowed.

Name: ______________________________

Question	Points	Score
1	15	
2	10	
3	10	
4	10	
5	10	
6	15	
Total:	70	

1. (a) (5 points) For each of the following, give a simpler description the module up to isomorphism (e.g. $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}/15\mathbb{Z}$). Do not justify your answers.

$$\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \cong$$

$$\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/5\mathbb{Z} \cong$$

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^2 \cong$$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}) \cong$$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{Z}/3\mathbb{Z}) \cong$$

(b) (10 points) Suppose that V and W are vectorspaces over a field F of dimensions n and m, respectively. For each of the following F-vectorspaces give the dimension and a basis, assuming that $\{v_1, \ldots, v_n\}$ is a basis for V and $\{w_1, \ldots, w_m\}$ is a basis for W. Do not justify your answers.

Module	Dimension	Basis
$V \oplus W$		
$V \otimes_F W$		
$\mathrm{Hom}_F(V, W)$		
$\mathcal{S}^3(V)$		
$\bigwedge(V)$		

2. Let A, B, and C be R-modules.

   (a) (5 points) Give an explicit isomorphism

   $$\Phi : (A \oplus B) \otimes_R C \to (A \otimes_R C) \oplus (B \otimes_R C)$$

   by giving values of Φ and its inverse on simple tensors.
   Do not justify your answers.

   $\Phi((a, b) \otimes c)) =$

   $\Phi^{-1}(a \otimes c_1, b \otimes c_2) =$

   (b) (5 points) Similarly, describe an explicit isomorphism

   $$\mathrm{Hom}_R(A \oplus B, C) \to \mathrm{Hom}_R(A, C) \oplus \mathrm{Hom}_R(B, C)$$

   and its inverse. Do not justify your answers.

3. (10 points) Given an R-module M, the set of torsion elements of M is defined to be

$$\mathrm{Tor}(M) = \{m \in M : rm = 0 \text{ for some } r \in R \backslash \{0\}\}.$$

Show that if R is an integral domain, then $\mathrm{Tor}(M)$ is a submodule of M.

4. (10 points) Let $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ be an exact sequence of R-modules that splits. Prove that $B \cong A \oplus C$.

5. Let R denote the ring $\mathbb{Z} \oplus \mathbb{Z}$ under coordinate-wise addition and multiplication.

(a) (5 points) Show that $M = \mathbb{Z}$ is an R-module with $(a, b) \cdot z = az$.

(b) (5 points) Circle all that apply. Do not justify your answers.

As an R-module, $M = \mathbb{Z}$ is ...

free projective injective flat finitely generated

6. (15 points) True or false. Circle one. Do not justify your answers.

(a) An R-module M is finitely-generated if and only if it is a quotient of a free module R^n for some $n \in \mathbb{Z}_{\geq 0}$.

True False

(b) An R-module homomorphism $\Phi : M \otimes_R M \to N$ is injective if and only if the R-bilinear map $\phi : M \times M \to N$ given by $\phi(m_1, m_2) = \Phi(m_1 \otimes m_2)$ is injective.

True False

(c) If R is a commutative ring with $1_R \neq 0$, then every R-algebra is commutative.

True False

(d) For every R-module homomorphism $\varphi : A \to B$ there are submodules $A' \subseteq A$ and $B' \subseteq B$ for which the sequence $0 \to A' \xrightarrow{i} A \xrightarrow{\varphi} B' \to 0$ is exact, where the map $i : A' \to A$ is given by inclusion.

True False

(e) $\langle x^2 \rangle$ is a graded ideal in $\mathbb{Q}[x, y]$.

True False