MA 721 – Final Exam

Spring 2020

Time: 3 hours

1. Write out solutions to all questions on your own paper and send me an electronic copy by email or on Moodle by **8am on Wednesday, April 29, 2020**.
2. You do not need to rewrite the problem statements, but make sure that all problems are clearly labeled and in order.
3. Unless stated otherwise, **justify your answers** to receive full credit. Your answers do not have to be in complete sentences, but they do need to be understandable.
4. **No outside assistance** (from notes, books, internet, people, etc.) is allowed.
5. You will have **3 hours** to write up solutions, starting when you look at any page after this one. (This does not include scanning/sending time.)
6. Please **write out** the following on the top of your exam solutions and **sign** your name after it:

   I affirm that I have not used any outside resources while taking this exam and that I have completed the exam in a continuous 3-hour time period.

Question	Points	Score
1	10	
2	5	
3	9	
4	10	
5	10	
6	15	
7	10	
8	20	
9	21	
Total:	110	

1. (10 points) Consider the ring $R = M_2(\mathbb{Z})$ of 2×2 matrices over $\mathbb{Z}$. Let $M = \mathbb{Z}^2$.

   (a) Show that M is an R-module under the action of left multiplication (i.e. identifying M with the space of 2×1 matrices over $\mathbb{Z}$ and letting $A \cdot v = Av$ for $A \in R$, $v \in M$).

   (b) Find a nontrivial submodule $0 \subsetneq N \subsetneq M$.

2. (5 points) Find the Jordan canonical form of the linear transformation given by multiplication by $1 + x$ on the $\mathbb{Q}$-vectorspace $\mathbb{Q}[x]/\langle x^2(x-1)^2 \rangle$. Do not justify your answers.

3. (9 points) Fill in the blank with another commonly used term from algebra (not involving the term "module"). Do not justify your answers.

   (a) A module over a field F is the same as a(n) ______________.

   (b) A module over $\mathbb{Z}$ is the same as a(n) ______________.

   (c) An R-submodule of a ring R is the same as a(n) ______________.

4. (10 points) Let $R = M_{n_1}(\mathbb{C}) \times \ldots \times M_{n_r}(\mathbb{C})$, where $n_1, \ldots, n_r \in \mathbb{Z}_{>0}$.
   In terms of r, $n_1, \ldots, n_r$, describe each of the following. Do not justify your answers.

   (a) The dimension of R as a $\mathbb{C}$-vectorspace.

   (b) The dimension of the center of R, as a $\mathbb{C}$-vectorspace.

   (c) The number of primitive central idempotents in R.

   (d) The number of distinct (i.e. non-isomorphic) irreducible R-modules.

   (e) The dimensions of the distinct irreducible R-modules.
   (This should be a list of numbers with length equal to your answer from (d).)

5. (10 points) Let G be a finite group and $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ a representation of G with character χ. Suppose that $g \in G$ has order 2. Show that $\chi(g)$ is an integer.

   (Hint: what are the possibilities for the minimal polynomial and eigenvalues of $\varphi(g)$?)

6. (15 points) Here is the character table of a mystery finite group G, with conjugacy classes $\{\mathrm{id}\}, K_2, K_3, K_4$:

	$\{\mathrm{id}\}$	K_2	K_3	K_4
χ_1	1	1	1	1
χ_2	1	1	1	-1
χ_3	2	$(-1+\sqrt{5})/2$	$(-1-\sqrt{5})/2$	0
χ_4	2	$(-1-\sqrt{5})/2$	$(-1+\sqrt{5})/2$	0

Make sure to justify your answers to each of the following:

(a) What is the size of G?

(b) Is G abelian?

(c) What are the sizes of the conjugacy classes K_2, K_3, K_4?

(d) Let φ be a representation of G with character χ where $\chi(g) = \chi_3(g)^2\chi_4(g)^2$ for all $g \in G$, i.e.

	$\{\mathrm{id}\}$	K_2	K_3	K_4
χ	16	1	1	0

How does φ decompose into irreducible representations of G?

7. (10 points) Let F be the field $\mathbb{Z}/2\mathbb{Z}$. Given a set A, let FA denote the free F-module on A (i.e. the F-vectorspace with basis A). Consider the maps

$$0 \to F\{f_{123}, f_{124}\} \xrightarrow{d_1} F\{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\} \xrightarrow{d_2} F\{v_1, v_2, v_3, v_4\} \xrightarrow{d_3} 0 \to \dots$$

given by

$$d_1(f_{ijk}) = e_{ij} + e_{ik} + e_{jk} \quad \text{and} \quad d_2(e_{ij}) = v_i + v_j.$$

(a) Show that this is a cochain complex $\mathcal{C}$.

(b) Find the dimension, as an F-vectorspace, of $H^0(\mathcal{C})$, $H^1(\mathcal{C})$, and $H^2(\mathcal{C})$.

(Hint: first calculate the dimensions of the kernels and images of each d_i, and remember the rank-nullity theorem from linear algebra. You can use without proof that the image of d_2 is the (3-dimensional) hyperplane $\{a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 : a_1 + a_2 + a_3 + a_4 = 0\}$.)

8. (20 points) Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and the sequence of maps

$$0 \xrightarrow{\phi_2} \mathbb{Z}^2 \xrightarrow{\phi_1} \mathbb{Z}^3 \xrightarrow{\phi_0} M \to 0$$

where $\phi_2(0) = (0,0)$, $\phi_1(a,b) = (0, 2a, 3b)$ and $\phi_0(x,y,z) = (x, y + 2\mathbb{Z}, z + 3\mathbb{Z})$.

(a) Show that this is a projective resolution of M as a $\mathbb{Z}$-module.

(b) Describe the cochain complex and maps obtained from applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2\mathbb{Z})$ to this resolution and use it to calculate $\mathrm{Ext}^n_R(M, \mathbb{Z}/2\mathbb{Z})$ for all $n \geq 0$.

(c) Describe the chain complex and maps obtained from applying $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} -$ to this resolution and use them to calculate $\mathrm{Tor}^R_n(\mathbb{Z}/2\mathbb{Z}, M)$ for all $n \geq 0$.

(Your final answers to (b) and (c) should be expressions as finite abelian groups, not involving quotients, "ker", or "image")

9. (21 points) For each, answer **True** or **False**. Do not justify your answer.

(a) For any submodule N of a module M, there exists another submodule N' of M so that $M = N \oplus N'$.

(b) The tensor algebra, $\mathcal{T}(M)$, of a module M over a commutative ring R is commutative.

(c) Every matrix $A \in M_n(\mathbb{C})$ is similar to one of the form $N + D$ where N is nilpotent and D is diagonal.

(d) Every finitely-generated torsion-free module over a PID is free.

(e) The module $\mathbb{Q}[x]/\langle x \rangle \oplus \mathbb{Q}[x]/\langle x - 1 \rangle$ can be generated (as a $\mathbb{Q}[x]$-module) by a single element.

(f) Every radical ideal in a Noetherian ring is the intersection of finitely many prime ideals.

(g) Let G be a finite group. Every function $G \to \mathbb{C}$ that is constant on conjugacy classes is the character of some representation of G.