Today: Characters of S_4 (§19.1) and S_n

Mon 4/27: OH 1-3pm

Tues 4/28: Final Exam

Closed notes/book etc.

3 hours to take in a 24-hour window

Email me solutions or submit on Moodle

Cumulative, slight emphasis on Ch 17, 18.

Last time: hunting for irreducible reps/characters of S_4

$\# = \#$ conjugacy classes $= 5$

dimensions: $n_1 = n_2 = 1, \quad n_3 = 2, \quad n_4 = n_5 = 3$

1-dim'l reps: trivial and sign

Perm rep: $\varphi : S_4 \to GL_4(\mathbb{C}) \quad \varphi(\pi) = P_\pi$ (perm matrix)

character $\rho(\pi) = \text{trace}(P_\pi) = \#$ fixed pts of π

Invariant subspaces: $V = \text{span}_{\mathbb{C}}\{(1,1,1,1)\}$

(1-dim'l rep, V fixed by $P_\pi \to$ trivial rep)

$V^\perp = \{x \in \mathbb{C}^4 : \sum_{i=1}^{4} x_i = 0\}$

$\to$ (3-dim'l rep of S_4)

What is its character?

Character of φ

$= (\text{char. of } \varphi|_V) + (\text{char. of } \varphi|_{V^\perp})$

$\Rightarrow \quad \chi = \text{char of } \varphi|_{V^\perp} = \rho - \chi_{\text{trivial}}$

$\chi(\pi) = \rho(\pi) - \chi_{\text{trivial}}(\pi) = \#\text{fixed pts of } \pi - 1.$

$$\langle \chi, \chi \rangle = \frac{1}{24}\left(\chi(id)^2 + 6\chi((12))^2 + 8\chi((123))^2 + 6\chi((1234))^2 + 3\chi((12)(34))^2\right)$$

$$= \frac{1}{24}\left(3^2 + 6\cdot 1^2 + 8\cdot 0^2 + 6\cdot(-1)^2 + 3\cdot(-1)^2\right)$$

$$= \frac{1}{24}(9+6+6+3) = 1.$$

Prop: A character χ of G is irreducible $\Leftrightarrow$ $\langle \chi, \chi \rangle = 1$.

$(\Rightarrow)$ Irred. char. form an orthnormal basis $\Rightarrow$ $\langle \chi, \chi \rangle = 1$.

$(\Leftarrow)$ $\chi = \sum_{i=1}^{r} a_i \chi_i$, $a_i \in \mathbb{Z}_{\geq 0}$

$$\langle \chi, \chi \rangle = \left\langle \sum_{i=1}^{r} a_i \chi_i, \sum_{j=1}^{r} a_j \chi_j \right\rangle = \sum_{1 \leq i,j \leq r} a_i \bar{a}_j \langle \chi_i, \chi_j \rangle$$

$$= \sum_{i=1}^{r} a_i^2$$

If this equals 1, $a_i = 1$ for some i and $a_j = 0$ for all $j \neq i$.

$\Rightarrow$ $\chi = \chi_i$.

2-dim'l rep of S_4 $\quad V = \{id, (12)(34), (13)(24), (14)(23)\}$

$V \setminus \{id\}$ is a conj. class in S_4.

Acting on $V \setminus \{id\}$ by conjugation gives homomorphism $S_4 \to S_{V \setminus \{id\}} \cong S_3$

$\pi = (12)$ $\quad (12)(34)$ $\quad (13)(24)$ $\quad (14)(23)$ $\quad (12) \mapsto (23)$

Kernel of hom $= V$.

From 2-dim'l rep. of S_3, we get $S_4 \to S_3 \to GL_2(\mathbb{C})$.

$\Rightarrow$ 2-dim'l rep of S_4

S_4		S_3	character
id	$\longrightarrow$	id	2
(12)	$\longrightarrow$	(23)	0
(123)	$\longrightarrow$	(132)	-1
(1234)	$\longrightarrow$	(13)	0
(12)(34)	$\longrightarrow$	id	2

Check irreducible by computing $\langle \chi, \chi \rangle$.

Character table for S_4

$1^2+(-1)^2+0^2+1^2+a^2 = \frac{|G|}{|K|} = \frac{24}{6} = 4$

$\Rightarrow a^2=1$

(2nd orth. relation)

	id	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	1	0	-1	-1
χ_5	3	$\boxed{-1}$	0	1	-1

$\leftarrow$ product of χ_2 and χ_4

Use $\langle \chi_i, \chi_5 \rangle = 0$ for $i=1,..,4$ to find χ_5

Prop: If V, W are $\mathbb{C}G$-modules with characters χ, τ (resp.) then character of $V \otimes_{\mathbb{C}G} W$ is $\chi\tau$ where

$$\chi\tau(g) = \chi(g)\tau(g).$$

(Proof) Pick basis $\{v_1,..,v_n\}$ for V and $\{w_1,..,w_m\}$ for W.

Then $\{v_1 \otimes w_1, v_2 \otimes w_1, .., v_n \otimes w_1, \dots, v_n \otimes w_m\}$ is a basis for $V \otimes W$.

Let φ, ψ be reps. corresponding to V, W.

W.r.t. these bases $\varphi(g) = A$, $\psi(g) = B = (b_{ij})_{ij}$.

$$\varphi \otimes \psi(g) = \begin{pmatrix} b_{11}A & \dots & b_{1m}A \\ b_{21}A & & \vdots \\ \vdots & & \vdots \\ b_{m1}A & \dots & b_{mm}A \end{pmatrix}$$

trace $= \sum_{i=1}^{m} b_{ii}\,\text{trace}(A)$

$= \text{trace}(A)\,\text{trace}(B).$

$\square$

Rep. Theory of S_n

$\downarrow$ not on exam

Good ref: "The Symmetric Group" by Bruce Sagan

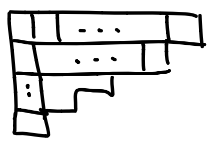
\# irred. rep. of S_n = \# conjugacy classes in S_n

= \# partitions of n $\lambda = (\lambda_1, \dots, \lambda_\ell)$

$\lambda_1 \geq \dots \geq \lambda_\ell \in \mathbb{Z}_{>0}$, $\sum_{i=1}^{\ell} \lambda_i = n$ (write $\lambda \vdash n$)

Ex: ($n=4$) (4), $(3,1)$, $(2,2)$, $(2,1,1)$, $(1,1,1,1)$

Shape corresponding to λ



$\leftarrow \lambda_1$ boxes

$\leftarrow \lambda_2$ boxes

$\vdots$

$\leftarrow \lambda_\ell$ boxes

(n boxes total)

A Young tableau is a filling of $\uparrow$ with the numbers $1, \dots, n$.

A Standard Young tableau is a Young tableau whose rows strictly increase ($\rightarrow$), cols. increase ($\downarrow$)

Let f_λ be the \# of standard. Young tableau

Ex: ($n=4$)

$\lambda = (4)$ $\quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}$ $\quad f_\lambda = 1$

$\lambda = (1,1,1,1)$ $\quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}$ $\quad f_\lambda = 1$

$\lambda = (2,2)$ $\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ $\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$ $\quad f_\lambda = 2$

$\lambda = (3,1)$ $\quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$ $\quad f_\lambda = 3$

$\lambda = (2,1,1)$ $\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$ $\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$ $\quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ $\quad f_\lambda = 3$

$\lambda \vdash n$ (λ is a partition of n)

Thm: The irred. $\mathbb{C}S_n$-modules are $\{S^\lambda : \lambda \vdash n\}$ "Specht modules"
where $\dim_{\mathbb{C}}(S^\lambda) = f_\lambda$.

Cor: $\sum_{\lambda \vdash n} f_\lambda^2 = n!$

Explicitly given a Young tableau of shape λ, define subgroups of S_n

$$R_T = \text{subgroup stabilizing rows of } T \quad \cong S_{\lambda_1} \times \dots \times S_{\lambda_\ell}$$

$$C_T = \quad " \qquad " \qquad \text{cols of } T$$

Young symmetrizer $y_T = \sum_{\substack{\pi \in R_T \\ \sigma \in C_T}} \text{sign}(\sigma)\, \pi\sigma \in \mathbb{C}S_n$

Then $S^\lambda =$ cyclic module $(\mathbb{C}S_n) y_T$.

Ex: $T = \boxed{1 \mid 2 \mid \cdots \mid n}$ $\quad R_T = S_n \quad C_T = \{id\}$

$$y_T = \sum_{\pi \in S_n} \pi \in \mathbb{C}S_n$$

$(\mathbb{C}S_n) y_T = \mathbb{C} y_T \Rightarrow$ trivial rep of S_n

Ex: $T = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}$ $\quad R_T = \{id\} \quad C_T = S_n$

$$y_T = \sum_{\sigma \in S_n} \text{sign}(\sigma)\, \sigma \in \mathbb{C}S_n$$

$(\mathbb{C}S_n) y_T = \mathbb{C} y_T \Rightarrow$ sign rep of S_n.