Today: Properties & examples of character tables (DF 18.3, 19.1)

Final exam: Tues 4/28, 3 hr take home
(Details on course website)

OH: W 4/22 1:30-2:30pm (or by appointment)
M 4/27 1-3pm

INNER PRODUCT ON CHARACTERS

G = finite group, Conjugacy classes $K_1, \ldots, K_r$

Inner product: $\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)} = \frac{1}{|G|} \sum_{i=1}^{r} |K_i| \chi(g_i) \overline{\psi(g_i)}$

where $g_i \in K_i$

First orthogonality relation: $\chi_1, \ldots, \chi_r$ irred. characters of G (over $\mathbb{C}$)

$\Rightarrow \langle \chi_i, \chi_j \rangle = \delta_{ij}$

That is, $\chi_1, \ldots, \chi_r$ are orthonormal w.r.t. $\langle \, , \rangle$.

Cor: We can use $\langle \, , \chi_i \rangle$ to decompose an arbitrary rep. of G into irreducibles.

M = a $\mathbb{C}G$-module, character χ

$M \cong a_1 M_1 \oplus \ldots \oplus a_r M_r$

where $M_1, \ldots, M_r$ = irred. $\mathbb{C}G$-modules

$a_i = \underline{\langle \chi, \chi_i \rangle} \in \mathbb{Z}_{\geq 0}$

Ex: $G = \langle \sigma : \sigma^n = 1 \rangle \Rightarrow n$ irred. rep. of dim 1

$\chi_j(\sigma) = \omega^j$ where $\omega = e^{2\pi i/n}$

$$\langle \chi_i, \chi_j \rangle = \frac{1}{n} \sum_{k=0}^{n-1} \chi_i(\sigma^k) \overline{\chi_j(\sigma^k)} = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{ik} \omega^{-jk}$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} (\omega^{i-j})^k = \frac{1}{n}(1 + x + \ldots + x^{n-1}) \Big|_{x = \omega^{i-j}} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

A 2-dim'l rep of G: $\varphi: G \to GL_2(\mathbb{C})$ $\varphi(\sigma) = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$

Character $\chi(\sigma^k) = 2\cos(2\pi k/n)$

$= \omega^k + \bar{\omega}^k$ $\quad \omega = e^{2\pi i/n}$

$\Rightarrow \chi = \chi_1 + \chi_{n-1}$

Indeed for $U = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$, $U^{-1}\varphi(\sigma^k)U = \begin{pmatrix} \omega^k & 0 \\ 0 & \bar{\omega}^k \end{pmatrix}$.

Def The character table of G is an $r \times r$ table with rows indexed by irred. reps. $\chi_1, \ldots, \chi_r$ of G and columns indexed by conjugacy classes $K_1, \ldots, K_r$ of G. The (i,j)th entry is $\chi_i(g)$ for $g \in K_j$.

Ex: $G = \langle \sigma : \sigma^3 = 1 \rangle$

character table for G $\longrightarrow$

	1	σ	σ^2
χ_1	1	ω	ω^2
χ_2	1	ω^2	ω
χ_3	1	1	1

The Second Orthogonality Relation for Characters:

If $\chi_1, \ldots, \chi_r$ are the irred. characters of G, $g, h \in G$

$$\sum_{i=1}^{r} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} |C_G(g)| & \text{if } g, h \text{ conjugate} \\ 0 & \text{o.w.} \end{cases}$$

where $C_G(g) = \{x \in G : xg = gx\}$, and $|C_G(g)| = |G| / |K_g|$.

K_g: conj. class of g.

(Proof) let T be the $r \times r$ character table of G.

Matrix version of first orthog. relation:

$$T \begin{pmatrix} |K_1| & & \\ & \ddots & \\ & & |K_r| \end{pmatrix} T^* = \left(\sum_{\ell=1}^{r} |K_\ell| \chi_i(g_\ell)\overline{\chi_j(g_\ell)} \right)_{ij} = |G| I_r.$$

Mult on left by T^{-1} $\curvearrowright$ $\begin{pmatrix} |K_1| & & \\ & \ddots & \\ & & |K_r| \end{pmatrix} T^* = |G| \cdot T^{-1}$

Mult on left by $\begin{pmatrix} |K_1| & & \\ & \ddots & \\ & & |K_r| \end{pmatrix}^{-1}$ $\curvearrowright$ $T^* T = |G| \cdot \begin{pmatrix} 1/|K_1| & & \\ & \ddots & \\ & & 1/|K_r| \end{pmatrix}.$ Mult on right by T

$\square$

Ex: $G = S_3$

$$T = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{ccc} \text{id} & (12) & (123) \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{pmatrix} \end{array}$$

$$\sum_{i=1}^{3} \chi_i(g)\overline{\chi_i(g)} = \begin{cases} 1^2 + 1^2 + 2^2 = 6 = \frac{6}{1} & \text{for } g = \text{id} \\ 1^2 + (-1)^2 + 0^2 = 2 = \frac{6}{3} & \text{for } g = (12) \\ 1^2 + 1^2 + (-1)^2 = 3 = \frac{6}{2} & \text{for } g = (123) \end{cases}$$

Conjugacy classes in S_4:

representative :	id	(12)	(12)(34)	(123)	(1234)
size :	1	6	3	$8 = \binom{4}{3} \cdot 2!$	$6 = 3!$

One familiar representation: $\pi \mapsto P_\pi \in GL_4(\mathbb{C})$ (group hom)

$\pi \mapsto \text{trace}(P_\pi) \in \mathbb{C}$ (character)

	(id)	(12)	(12)(34)	(123)	(1234)
perm rep	$\boxed{4}$	2	0	1	0

dim of rep

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1 dim'l reps of S_4: $\varphi : S_4 \to GL_1(\mathbb{C}) = \mathbb{C}^*$ (group hom)

character: $\pi \to \text{trace}(\varphi(\pi)) = \varphi(\pi)$

Note $\varphi(\text{id}) = 1$. Because φ is a group hom.

$$1 = \varphi(\text{id}) = \varphi((12)(12)) = \varphi(12)\varphi(12) \quad \Rightarrow \varphi(12) = \pm 1$$

If $\pi \in S_4$ is a product of transpositions, $\pi = t_1 \cdots t_s$ then $\varphi(\pi) = \varphi(t_1)\cdots\varphi(t_s) = \varphi((12))^s$.

$\Rightarrow$ Only 2 irred. 1-dim'l reps of S_4: trivial and sign

	(id)	(12)	(12)(34)	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1

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5 irred. reps total of dimensions $n_1 \le n_2 \le \ldots \le n_5$.

Know $n_1 = n_2 = 1$ and $2 \le n_3 \le n_4 \le n_5$.

Also $n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = |G| = 24$

Only possibility is $n_3 = 2$, $n_4 = n_5 = 3$.