Today: §18.3 Character Theory

Next week: Practice and Examples

PART I

G = finite group r = # conjugacy classes of G

Def: A <u>class function</u> on G is a function $\theta: G \to \mathbb{C}$ that is constant on conjugacy classes, i.e.

$$\theta(h^{-1}gh) = \theta(g) \quad \forall\, h, g \in G.$$

Remark: The set of class functions is a $\mathbb{C}$-vectorspace of dim r. It contains all the characters of G.

Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G.

Last time: $\chi_1, \ldots, \chi_r$ are linearly indep. over $\mathbb{C}$

$\Longrightarrow$ $\chi_1, \ldots, \chi_r$ form a basis for $\mathbb{C}$-vec. space of class functions!

Today: Define an inner product w.r.t. which $\chi_1, \ldots, \chi_r$ are orthonormal

Def: Given class functions θ, τ, define

$$\langle \theta, \tau \rangle = \frac{1}{|G|} \sum_{g \in G} \theta(g) \overline{\tau(g)}.$$

complex conjugation

Claim: $\langle\, ,\, \rangle$ is a positive definite Hermitian inner product on the $\mathbb{C}$-vec. space of class functions.

1) $\langle \tau, \theta \rangle = \overline{\langle \theta, \tau \rangle}$ Check!

2) $\langle a\theta_1 + b\theta_2, \tau \rangle = a\langle \theta_1, \tau \rangle + b\langle \theta_2, \tau \rangle \quad \forall a, b \in \mathbb{C}$

3) $\langle \theta, \theta \rangle \geq 0$ with $\langle \theta, \theta \rangle = 0 \iff \theta = 0.$

Remark: Let $K_1, \ldots, K_r$ be the conjugacy classes of G, with representatives $x_1, \ldots, x_r$ (where $x_i \in K_i$).

Then

$$\langle \theta, \tau \rangle = \frac{1}{|G|} \sum_{g \in G} \theta(g)\overline{\tau(g)} = \frac{1}{|G|} \sum_{i=1}^{r} |K_i|\, \theta(x_i)\overline{\tau(x_i)}.$$

Ex: $G = S_3$ $\quad r = 3 \quad K_1 = \{id\} \quad K_2 = \{(12), (13), (23)\} \quad K_3 = \{(123), (132)\}$

irred. characters

character table of S_3

	K_1	K_2	K_3
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$$\begin{aligned}
\langle \chi_1, \chi_2 \rangle &= \frac{1}{6}\left(\chi_1(id)\overline{\chi_2(id)} + \chi_1((12))\overline{\chi_2((12))} + \ldots + \chi_1((132))\overline{\chi_2((132))}\right) \\
&= \frac{1}{6}\left(\chi_1(id)\overline{\chi_2(id)} + 3\chi_1((12))\overline{\chi_2((12))} + 2\chi_1((123))\overline{\chi_2((123))}\right) \\
&= \frac{1}{6}\left(1 \cdot 1 + 3 \cdot 1 \cdot (-1) + 2 \cdot 1 \cdot 1\right) \\
&= \frac{1}{6}(1 - 3 + 2) = 0.
\end{aligned}$$

Thm (First Orthogonality Relation for Group Characters) **PART II**

W.r.t. the inner product $\langle \, , \rangle$, $\chi_1, \ldots, \chi_r$ are orthonormal.

That is

$$\langle \chi_i, \chi_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\varphi_1, \ldots, \varphi_r$ be the irreducible representations of G.

$\chi_1, \ldots, \chi_r$ " " characters "

$M_1, \ldots, M_r$ " " $\mathbb{C}G$-modules

$z_1, \ldots, z_r$ " primitive central idempotents of $\mathbb{C}G$.

Let $\rho : G \to \mathbb{C}$ denote the character of the regular representation of G

rep. corresponding to the $\mathbb{C}G$-module $\mathbb{C}G$

<u>Recall</u>: $\rho(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{o.w.} \end{cases}$ and $\rho = \sum_{j=1}^{r} \chi_j(1)\chi_j$

<u>Prop</u>: $$z_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \chi_i(g^{-1})\, g$$

(Proof) Write $z_i = \sum_{g \in G} \alpha_g g \in \mathbb{C}G$. Want to compute coefficient α_g

Applying ρ to $z_i g^{-1}$ and using linearity over $\mathbb{C}$ gives

$$\rho(z_i g^{-1}) = \rho\Big(\sum_{h \in G} \alpha_h h g^{-1}\Big) = \sum_{h \in G} \alpha_h \rho(h g^{-1}) = \alpha_g |G|$$

Using $\rho = \sum_{j=1}^{r} \chi_j(1)\chi_j$ gives

$$\rho(z_i g^{-1}) = \sum_{j=1}^{r} \chi_j(1) \underline{\chi_j(z_i g^{-1})}.$$

Note: $$\chi_j(z_i g^{-1}) = \operatorname{trace}(\varphi_j(z_i g^{-1})) = \operatorname{trace}(\boxed{\varphi_j(z_i)}\,\varphi_j(g^{-1}))$$

$$\boxed{\varphi_j(z_i)} = \begin{cases} I_{n_i} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$= \begin{cases} \chi_i(g^{-1}) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\Rightarrow \quad \alpha_g = \frac{\rho(z_i g^{-1})}{|G|} = \frac{1}{|G|} \sum_{j=1}^{r} \chi_j(1)\chi_j(z_i g^{-1}) = \frac{1}{|G|} \chi_i(1)\chi_i(g^{-1}).$$

$\square$

<u>IDEA</u>: Use orthonormality of $z_1, \dots, z_r$ to show orthonormality of "dual basis" $\chi_1, \dots, \chi_r$.

Lemma: If χ is a character of G, then $\chi(g^{-1}) = \overline{\chi(g)}$.

(Proof) Let $\varphi: G \to GL(V)$ be a rep. of G with deg n, char χ.

Take $g \in G$ and let $k = \mathrm{ord}(g)$.

$g^k = 1 \Rightarrow \varphi(g)^k = \mathrm{id}_V \Rightarrow$ min. poly. of $\varphi(g)$ divides $x^k - 1$.

$\Rightarrow$ roots of min. poly. are distinct elt. of $\{\omega^l : l = 0, \ldots, k-1\}$

where $\omega = e^{2\pi i/k}$

$\Rightarrow$ w.r.t. some basis of V

$$\varphi(g) = \begin{pmatrix} \omega^{l_1} & & \\ & \ddots & \\ & & \omega^{l_n} \end{pmatrix} \Rightarrow \varphi(g^{-1}) = \begin{pmatrix} \omega^{-l_1} & & \\ & \ddots & \\ & & \omega^{-l_n} \end{pmatrix}$$

$$\Rightarrow \chi(g^{-1}) = \sum_{i=1}^{n} \omega^{-l_i} = \sum_{i=1}^{n} \overline{\omega^{l_i}} = \overline{\sum_{i=1}^{n} \omega^{l_i}} = \overline{\chi(g)}. \qquad \square$$

(Proof of First Orthogonality Theorem)

Let $1 \le i, j \le r$.

$$\delta_{ij} z_i = z_i z_j = \frac{\chi_i(1)\chi_j(1)}{|G|^2} \sum_{g,h \in G} \chi_i(g^{-1})\chi_j(h^{-1}) \underbrace{gh}_{\substack{=1 \\ \Leftrightarrow\, g^{-1} = h}}$$

$\Rightarrow$ coeff of $1 = \mathrm{id}_G$ in $\delta_{ij} z_i$ is $\frac{\chi_i(1)\chi_j(1)}{|G|^2} \sum_{h \in G} \chi_i(h) \underbrace{\chi_j(h^{-1})}_{\substack{\parallel \text{ (Lemma)} \\ \overline{\chi_j(h)}}}$

$$= \frac{\chi_i(1)\chi_j(1)}{|G|} \langle \chi_i, \chi_j \rangle$$

By formula for z_i in Prop,

coeff of id_G in $\delta_{ij} z_i$ is $\delta_{ij} \frac{\chi_i(1)^2}{|G|}$

$$\Rightarrow \langle \chi_i, \chi_j \rangle = \delta_{ij} \cdot \frac{\chi_i(1)}{\chi_j(1)} = \delta_{ij}.$$

$\square$

PART III

Cor: For any class function θ, $\theta = \sum_{i=1}^{r} \langle \theta, \chi_i \rangle \chi_i$

(Proof) $\theta = \sum_{i=1}^{r} \alpha_i \chi_i \Rightarrow \langle \theta, \chi_j \rangle = \sum_{i=1}^{r} \alpha_i \langle \chi_i, \chi_j \rangle = \alpha_j$.

Cor: Let M be a $\mathbb{C}G$-module with character χ and irreducible decomposition $M \cong a_1 M_1 \oplus \dots \oplus a_r M_r$. Then $\chi = a_1\chi_1 + \dots + a_r \chi_r$ and $a_i = \langle \chi, \chi_i \rangle$.

Ex: $G = S_3$ $\quad \varphi : G \to GL_3(\mathbb{C})$ permutation rep.

χ = char of $\varphi \Rightarrow \chi(id) = 3 \quad \chi((12)) = 1 \quad \chi((123)) = 0$

$$\langle \chi, \chi_i \rangle = \frac{1}{6}\left[1 \cdot \chi(id) \cdot \overline{\chi_i(id)} + 3 \cdot \chi((12)) \cdot \overline{\chi_i((12))} + 2 \cdot \chi((123)) \overline{\chi_i((123))}\right]$$

$$\langle \chi, \chi_1 \rangle = \frac{1}{6}\left[1 \cdot 3 \cdot \overline{1} + 3 \cdot 1 \cdot \overline{1} + 2 \cdot 0 \cdot \overline{1}\right] = \frac{1}{6}[3+3+0] = 1$$

$$\langle \chi, \chi_2 \rangle = \frac{1}{6}\left[1 \cdot 3 \cdot \overline{1} + 3 \cdot 1 \cdot \overline{(-1)} + 2 \cdot 0 \cdot \overline{1}\right] = \frac{1}{6}[3-3+0] = 0$$

$$\langle \chi, \chi_3 \rangle = \frac{1}{6}\left[1 \cdot 3 \cdot \overline{2} + 3 \cdot 1 \cdot \overline{0} + 2 \cdot 0 \cdot \overline{(-1)}\right] = \frac{1}{6}[6+0+0] = 1$$

$\Rightarrow \chi = \chi_1 + \chi_3$

$\Rightarrow \varphi$ decomposes as $\varphi_1 \oplus \varphi_3$.