Today: start §18.3

Hwk 10 due Fri 4/17

Sat 4/18 = makeup Thurs – I will post some videos/notes
Please watch before Tues. class

Hwk 11 cancelled

(I will drop lowest 3 hwk grades)

Previously... for a finite group G,

$$\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C}) \quad \text{for some } n_1, \dots, n_r \in \mathbb{Z}_+$$

$$Z(\mathbb{C}G) \cong \{(a_1 I_{n_1}, \dots, a_r I_{n_r}) : a_1, \dots, a_r \in \mathbb{C}\} \cong \mathbb{C}^r$$

$r = \#$ conjugacy classes in G

Def: A <u>primitive central idempotent</u> of a ring R is an idempotent $e \in Z(R)$ that cannot be written as the sum of two orthogonal idempotents in $Z(R)$.

Remark: These need not be primitive idempotents!

e.g. in $M_2(\mathbb{C})$, $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is a primitive central idempotent.

not primitive, since $\begin{pmatrix}1&0\\0&1\end{pmatrix} = \underbrace{\begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix}}$

sum of two orth. idempotents (not in $Z(M_2(\mathbb{C}))$)

In $M_{n_1}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C})$, the primitive central idempotents are

$$\{(0, \dots, 0, I_{n_i}, 0, \dots, 0) : i = 1, \dots, r\}.$$

Let $z_1, \dots, z_r$ be the corresponding prim. cent. idempts of $\mathbb{C}G$.

Let $M_1, \dots, M_r$ be the irreducible $\mathbb{C}G$-modules.

(M_i obtained from action on any column of $M_{n_i}(\mathbb{C})$ component)

$\Rightarrow$ deg n_i

Since any $\mathbb{C}G$-module M is completely reducible,

$$M \cong a_1 M_1 \oplus \dots \oplus a_r M_r \quad \text{for some } a_1, \dots, a_r \in \mathbb{Z}_{\geq 0}$$

where $a_i M_i = \underbrace{M_i \oplus \dots \oplus M_i}_{a_i}$.

Prop: 1) $M \cong z_1 M \oplus \dots \oplus z_r M$

2) $z_i M \cong \underbrace{M_i \oplus \dots \oplus M_i}_{a_i}$

(Proof) (1) Note $1 = z_1 + \dots + z_r \Rightarrow M = (z_1 + \dots + z_r) M$

$$= z_1 M + \dots + z_r M$$

Also $z_i z_j = 0$ for $i \neq j$.

If $m \in z_1 M$ $(m = z_1 x)$, then $z_1 m = z_1^2 x = z_1 x = m$.

If $m \in z_2 M + \dots + z_r M$ $(m = z_2 x_2 + \dots + z_r x_r)$, $z_1 m = 0$.

$\Rightarrow z_1 M \cap (z_2 M + \dots + z_r M) = \{0\}$.

(2) Note that $z_i M_i = M_i$ and $z_i M_j = 0$.

Cor: $M \cong a_1 M_1 \oplus \dots \oplus a_r M_r$ where $a_i = \dfrac{\dim_{\mathbb{C}}(z_i M)}{\dim_{\mathbb{C}}(M_i)}$

Ex: For $M = \mathbb{C}G$, $M \cong n_1 M_1 \oplus \dots \oplus n_r M_r$

$z_i M \cong M_{n_i}(\mathbb{C}) \leftarrow \dim n_i^2$ $\quad M_i \cong \mathbb{C}^{n_i} \leftarrow \dim n_i$

CHARACTERS OF REPRESENTATIONS

Def: Given a representation $\varphi : G \to GL_n(\mathbb{C})$, the <u>character of φ</u> is the function $\chi : G \to \mathbb{C}$ given by

$$\chi(g) = \operatorname{trace}(\varphi(g)).$$

The <u>degree of χ</u> is the degree of $\varphi = n = \chi(1_G)$
$= \operatorname{trace}(I_n)$.

We call χ <u>irreducible/reducible</u> if φ is irred./red.

Note: χ need not be a homomorphism of any kind.

Ex: (deg 1 rep) $\varphi : G \to GL_1(\mathbb{C}) \Rightarrow \chi(g) = \varphi(g) \quad \forall g \in G$.

Ex: (perm rep of S_n) $\varphi : S_n \to GL_n(\mathbb{C}) \quad \varphi(\pi) = P_\pi$ (perm matrix)

$$\chi(\pi) = \operatorname{trace}(P_\pi) = \#\text{fixed pts of } \pi$$

e.g. $n=3$

$\pi = id$	$\pi = (12)$	$\pi = (123)$
$\varphi(\pi) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$\varphi(\pi) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\varphi(\pi) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\chi(\pi) = 3$	$\chi(\pi) = 1$	$\chi(\pi) = 0$

Remark: χ is constant on conjugacy class of G

$$\operatorname{tr}(\varphi(hgh^{-1})) = \operatorname{tr}(\varphi(h)\varphi(g)\varphi(h)^{-1}) = \operatorname{tr}(\varphi(g)\varphi(h)\varphi(h)^{-1})$$
$$= \operatorname{tr}(\varphi(g))$$

$$\Rightarrow \chi(hgh^{-1}) = \chi(g)$$

Prop: The character of a direct sum of representations is the sum of the characters of each component.

(Proof idea) $\operatorname{trace}((\varphi_1 \oplus \varphi_2) g) = \operatorname{trace}\left(\begin{array}{c|c} \varphi_1(g) & 0 \\ \hline 0 & \varphi_2(g) \end{array}\right)$

$= \varphi_1(g) + \varphi_2(g)$.

Prop: Two representations of G are equivalent if and only if they have the same character.

(Proof) ($\Rightarrow$) φ, ψ equiv. rep. of G $\Rightarrow$ $\exists\, U \in GL_n(\mathbb{C})$ s.t.

$\psi(g) = U \varphi(g) U^{-1} \quad \forall g$.

$$\operatorname{tr}(\psi(g)) = \operatorname{tr}(U \varphi(g) U^{-1}) = \operatorname{tr}(\varphi(g) U U^{-1}) = \operatorname{tr}(\varphi(g)).$$

For ($\Leftarrow$) need:

Lemma: Let $\chi_1, \dots, \chi_r$ be the irreducible characters of G. Then $\chi_1, \dots, \chi_r$ are linearly indep. over $\mathbb{C}$.

(Proof of Lemma) By extending by linearity over $\mathbb{C}$, χ_i define $\mathbb{C}$-linear function on $\mathbb{C}G$.

If $z_1, \dots, z_r$ are the principal central idempotents in $\mathbb{C}G$,

$$\chi_i(z_j) = \begin{cases} n_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\Rightarrow$ $\chi_1, \dots, \chi_r$ are linearly indep.

If $\chi = \sum_{i=1}^{r} a_i \chi_i = 0 \Rightarrow \chi(z_j) = \sum_{i=1}^{r} a_i \chi_i(z_j) = a_i n_i = 0$

$\Rightarrow a_i = 0$ $\square$

($\Leftarrow$ of Prop) Suppose φ, ψ reps with same character χ.

Let M, N be the corresponding $\mathbb{C}G$-modules.

Then $\quad M \cong a_1 M_1 \oplus \dots \oplus a_r M_r$ for some $a_i \in \mathbb{Z}_{\geq 0}$

$\quad\quad N \cong b_1 M_1 \oplus \dots \oplus b_r M_r$ " $b_j \in \mathbb{Z}_{\geq 0}$

Then $\quad$ char. of $\varphi = a_1 \chi_1 + \dots + a_r \chi_r$

$\quad\quad$ char of $\psi = b_1 \chi_1 + \dots + b_r \chi_r$

By linear independence of $\chi_1, \dots, \chi_r$, $a_1 = b_1, \dots, a_r = b_r$.

$\Rightarrow \quad M \cong a_1 M_1 \oplus \dots \oplus a_r M_r = b_1 M_1 \oplus \dots \oplus b_r M_r \cong N$.

$M \cong N$ as $\mathbb{C}G$-modules $\Rightarrow \varphi, \psi$ equivalent

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Prop: Let $z_1, \dots, z_r$ be the principal central idempotents of $\mathbb{C}G$, and $\chi_1, \dots, \chi_r$ the irreducible characters, then

$$z_i = \frac{\chi_i(1_G)}{|G|} \sum_{g \in G} \chi_i(g^{-1})\, g.$$

Ex: $G = S_3 \quad r = 3 \quad |G| = 6$

(trivial rep) $\chi_1(\pi) = 1 \qquad z_1 = \frac{1}{6} \sum_{\pi \in S_3} \pi$

(sign rep) $\chi_2(\pi) = \text{sign}(\pi)$

$$z_2 = \frac{1}{6} \sum_{\pi \in S_3} \text{sign}(\pi)\, \pi$$

(deg 2 rep) $\chi_3(id) = 2$

$\chi_3((123)) = tr\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} = 2\cos(2\pi/3) = -1 \qquad \omega = e^{2\pi i/3}$

$\chi_3((23)) = tr\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$

$$z_3 = \frac{2}{6}\big(2\, id - (123) - (132)\big) = e_3 + e_4$$

where $e_3 = \frac{1}{3}\big(id + \omega(123) + \omega^2(132)\big)$, $e_4 = \frac{1}{3}\big(id + \omega^2(123) + \omega(132)\big)$