Today: finish §18.2
Thurs: start §18.3
Hwk 10 due Friday 4/17
Sat (4/18) = "make-up Thursday"

From last time:

Wedderburn's Thm: For a non-zero ring R with $1 \neq 0$, TFAE

1) Every R-module is projective
2) Every R-module is injective
3) Every R-module is completely reducible
4) As a left R-module $R \cong L_1 \oplus \dots \oplus L_n$ where $L_i = Re_i$ is simple and
$$e_i^2 = e_i, \quad e_i e_j = e_j e_i = 0, \quad \sum_{i=1}^{n} e_i = 1.$$
5) As a ring, $R = M_{n_1}(\Delta_1) \times \dots \times M_{n_r}(\Delta_r)$ where $n_1, \dots, n_r \in \mathbb{Z}_+$ and $\Delta_1, \dots, \Delta_r$ are division rings.

Remark: 1) Each $M_{n_i}(\Delta_i)$ factor contributes n_i isomorphic factors L_j to the direct sum in (4) (cols of matrix)

2) Every simple R-module is isomorphic to some L_j.

Important Example: FG = group ring of finite group G over F

Remark: For $R = \mathbb{C}G$, all division rings Δ_j in (5) are $\mathbb{C}$.

$\Rightarrow R \cong M_{n_1}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C})$

$\Rightarrow$ counting dim's over $\mathbb{C}$ gives $|G| = \sum_{i=1}^{r} n_i^2$

$$\#\text{conj. classes} = \dim\left(Z(\mathbb{C}G)\right) = r$$

Study by looking at FG as a FG-module. (corresponding rep. of G is called the <u>regular representation</u>)

<u>Ex</u>: $G = S_3$ $\quad R = \mathbb{C}G$

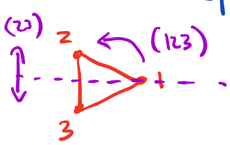
Conjugacy classes: $\{id\}$ $\{(12),(13),(23)\}$, $\{(123),(132)\}$ $\Rightarrow r = 3$

$n_1^2 + n_2^2 + n_3^2 = |G| = 6 \Rightarrow \quad n_1 = n_2 = 1, \; n_3 = 2$

Two irred. 1-dim'l rep.s : 1) $\varphi_1 : S_3 \to GL_1(\mathbb{C}) \quad \varphi_1(\pi) = 1$ (trivial rep)

2) $\varphi_2 : S_3 \to GL_1(\mathbb{C}) \quad \varphi_2(\pi) = \text{sign}(\pi) \in \{\pm 1\}$

2 dim'l rep: $\varphi_3 : S_3 \to GL_2(\mathbb{C})$



$$\varphi_3((123)) = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix} \qquad \varphi_3((23)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Over $\mathbb{C}$ we get an equivalent 2-dim'l rep. of S_3 by conjugating with $U = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$.

Gives

$$\tilde{\varphi}_3 : G \to GL_2(\mathbb{C}) \qquad \tilde{\varphi}_3((123)) = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} \quad \text{where } \omega = e^{2\pi i/3}$$

$$\tilde{\varphi}_3((23)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The map $S_3 \to GL_4(\mathbb{C})$ given by

$$\pi \longmapsto \begin{pmatrix} \varphi_1(\pi) & & \\ & \varphi_2(\pi) & \\ & & \tilde{\varphi}_3(\pi) \end{pmatrix}$$

induces ring isomorphism from $\mathbb{C}S_3$ to $M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_2(\mathbb{C})$.

$$(123) \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \omega^2 & \\ & & & \omega \end{pmatrix} \qquad (23) \mapsto \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

$$(12)+(13)+(23) = (123)(23) + (123)^2(23) + (23)$$

$$\mapsto \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \omega^2 \\ & & \omega & 0 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \omega \\ & & \omega^2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & & & \\ & -3 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}$$

Constant coeff over conj. classes $\Rightarrow$ in $Z(\mathbb{C}G)$

$\sum_{g \in G} \alpha_g g$ $\quad$ g, h conjugate $\Rightarrow \alpha_g = \alpha_h$

scalar $\times I_{n_i}$ in each factor $\Rightarrow$ in $Z(M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_2(\mathbb{C}))$

For explicit isomorphism:

$$e_1 = \frac{1}{6} \sum_{\pi \in S_3} \pi \qquad e_2 = \frac{1}{6} \sum_{\pi \in S_3} \operatorname{sign}(\pi)\, \pi$$

$$e_3 = \frac{1}{3}\left(\mathrm{id} + \omega(123) + \omega^2(132) \right) \qquad e_4 = \frac{1}{3}\left(\mathrm{id} + \omega^2(123) + \omega(132) \right)$$

Check: $e_i^2 = e_i$, $\quad e_i e_j = e_j e_i = 0$ for $i \neq j$ $\quad \sum_{i=1}^{4} e_i = \mathrm{id}$

$e_i \mapsto E_{ii}$ under ring isomorphism

Decomp. into simple modules:

$$L_1 = Re_1 = \mathbb{C}e_1 \qquad L_2 = Re_2 = \mathbb{C}e_2$$

$$L_3 = Re_3 = \operatorname{span}_{\mathbb{C}} \{ \mathrm{id} + \omega(123) + \omega^2(132),\ (12) + \omega(23) + \omega^2(13) \}$$

$$L_4 = Re_4 = \operatorname{span}_{\mathbb{C}} \{ \mathrm{id} + \omega^2(123) + \omega(132),\ (12) + \omega(13) + \omega^2(23) \}$$

As left $\mathbb{C}S_3$, $\mathbb{C}S_3 \cong L_1 \oplus L_2 \oplus L_3 \oplus L_4$ $\qquad$ $L_3 \cong L_4$

Over non-algebraically closed fields, numerology can be slightly different.

Ex: $G = \langle \sigma : \sigma^4 = 1 \rangle$ $F = \mathbb{R}$ $x^4 - 1 = (x-1)(x+1)(x^2+1)$

$$\mathbb{R}G \cong \mathbb{R}[x]/\langle x^4-1\rangle \cong \mathbb{R}[x]/\langle x-1\rangle \oplus \mathbb{R}[x]/\langle x+1\rangle \oplus \mathbb{R}[x]/\langle x^2+1\rangle$$
$$\cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C}$$
$$\cong M_1(\mathbb{R}) \times M_1(\mathbb{R}) \times M_1(\mathbb{C})$$

$r = 3$

$n_1 = n_2 = n_3 = 1$ $\sum_{i=1}^{3} n_i^2 = 3 \neq |G|$ $\sum_{i=1}^{3} n_i^2 \dim_{\mathbb{R}}(\Delta_i) = 1^2 + 1^2 + 1^2 \cdot 2$
$= |G| = 4$

Ex: $G = Q_8 = \langle i, j \mid i^4 = j^4 = 1,\ i^2 = j^2,\ j^{-1} i j = i^{-1} \rangle$

$= \{\pm 1, \pm i, \pm j, \pm k\}$ with $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$
$jk = -kj = i$
$ki = -ik = j$

$\mathbb{R}Q_8 \cong ?$ What are irred. rep of Q_8 over $\mathbb{R}$?

A 4-dim'l rep of Q_8 given by real Hamiltonian quaternions

$$\mathbb{H} = \{a + ib + jc + kd : a, b, c, d \in \mathbb{R}\}$$ (mult. given by mult in Q_8 and $\mathbb{R}$-linearity)

$$\cong \mathbb{R}Q_8 / \langle \boxed{1} + \boxed{-1} \rangle$$ $\boxed{\ }$ = group elt.

In $\mathbb{R}Q_8$, $\boxed{i} + \boxed{-i} = \boxed{i}(\boxed{1} + \boxed{-1})$ image in $\mathbb{H}$ is zero

Remark: $\mathbb{H}$ is a division ring!

Idempotents in $\mathbb{H}$? $e^2 = e,\ e \neq 0 \xRightarrow[\text{mult by } e^{-1}]{\text{div. ring}} e = 1$

only 1 idempotent

$\Rightarrow \mathbb{H}$ is an irreducible Q_8-module.

$\dim_{\mathbb{R}}(\mathbb{H}) = 4 \Rightarrow \mathbb{H}$ gives 4-dim'l representation of Q_8

w.r.t. basis $\{1, i, j, k\}$ we can write down homomorphism $Q_8 \to GL_4(\mathbb{R})$

$i \cdot 1 = i, \quad i \cdot i = -1, \quad i \cdot j = k, \quad i \cdot k = -j \qquad i \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ etc.

Other irred. rep's of Q_8 over $\mathbb{R}$:

1 dim'l rep's : $\varphi_1(i) = 1 \quad \varphi_1(j) = 1$ (trivial rep)

$\varphi_2(i) = -1 \quad \varphi_2(j) = 1$

$\varphi_3(i) = 1 \quad \varphi_3(j) = -1$

$\varphi_4(i) = -1 \quad \varphi_4(j) = -1$

$$\mathbb{R}Q_8 \cong M_1(\mathbb{R}) \times M_1(\mathbb{R}) \times M_1(\mathbb{R}) \times M_1(\mathbb{R}) \times M_1(\mathbb{H})$$

a noncomm. division ring

$r = 5 \qquad n_1 = \ldots = n_5 = 1$

$$\sum_{i=1}^{5} n_i^2 \dim_{\mathbb{R}}(\Delta_i) = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 \cdot 4 = 8 = |G|$$