Today: finish §18.2
Thurs: start §18.3
Hwk 10 due Friday 4/17
Sat (4/18) = "make-up Thursday"

From last time:

**Wedderburn's Thm:** For a non-zero ring $R$ with $1 \neq 0$, TFAE

1) Every $R$-module is projective
2) Every $R$-module is injective
3) Every $R$-module is completely reducible
4) As a left $R$-module $R \cong L_1 \oplus \cdots \oplus L_n$ where $L_i = Re_i$ is simple and $e_i^2 = e_i$, $e_i e_j = e_j e_i = 0$, $\sum_{i=1}^n e_i = 1$.
5) As a ring, $R = M_{n_1}(\Delta_1) \times \cdots \times M_{n_r}(\Delta_r)$ where $n_1, \ldots, n_r \in \mathbb{Z}_+$ and $\Delta_1, \ldots, \Delta_r$ are division rings.

**Remark:**
1) Each $M_{n_i}(\Delta_i)$ factor contributes $n_i$; isomorphic factors $L_j$ to the direct sum in (4) (cols of matrix).
2) Every simple $R$-module is isomorphic to some $L_j$.

**Important Example:** $FG =$ group ring of finite group $G$ over $F$

**Remark:** For $R = FG$, all division rings $\Delta_j$ in (5) are $\mathbb{C}$.

$\Rightarrow R \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$

$\Rightarrow$ counting dim's over $\mathbb{C}$ gives $\dim R = \sum_{i=1}^r n_i^2$

Number of conjugacy classes $= \dim (\mathbb{Z}(FG)) = r$
Study by looking at $FG$ as a $FG$-module. (corresponding rep. of $G$ is called the regular representation)

**Ex:** $G = S_3$, $R = \mathbb{C}G$

Conjugacy classes: $\{\text{id}\}$, $\{(12), (13), (23)\}$, $\{(123), (132)\} \Rightarrow r = 3$

$n_1^2 + n_2^2 + n_3^2 = |G| = 6 \Rightarrow n_1 = n_2 = 1$, $n_3 = 2$

Two irreducible $1$-dim'l reps:

1) $\psi_1: S_3 \to GL_1(\mathbb{C})$  
   $\psi_1(\pi) = 1$ (trivial rep)

2) $\psi_2: S_3 \to GL_1(\mathbb{C})$  
   $\psi_2(\pi) = \text{sign}(\pi) \in \{\pm 1\}$

Two dim'l reps:

$$\psi_3: S_3 \to GL_2(\mathbb{C})$$

$$\psi_3((123)) = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}$$  
   $\psi_3((23)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Over $\mathbb{C}$ we get an equivalent $2$-dim'l rep. of $S_3$ by conjugating with $U = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$.

Gives $\tilde{\psi}_3: G \to GL_2(\mathbb{C})$  
   $\tilde{\psi}_3((123)) = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$ where $\omega = e^{2\pi i/3}$

$\tilde{\psi}_3((23)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The map $S_3 \to GL_4(\mathbb{C})$ given by

$\pi \mapsto \begin{pmatrix} \psi_1(\pi) \\ \psi_2(\pi) \\ \tilde{\psi}_3(\pi) \end{pmatrix}$

induces a ring isomorphism from $\mathbb{C}S_3$ to $M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_2(\mathbb{C})$.

$(123) \mapsto \begin{pmatrix} 1 & \omega^2 \\ \omega & 1 \end{pmatrix}$  
   $(23) \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$
\[(12)+(13)+(23) = (123)(23) + (123)^2(23) + (23)\]

\[\uparrow\]

Constant coeff on conj. classes \[\Rightarrow \text{in } \mathbb{Z}(CG)\]

\[\sum_{g \in G} g, h \text{ conjugate} \Rightarrow \alpha_g = \alpha_h\]

For explicit isomorphism:

\[e_1 = \frac{1}{6} \sum_{\pi \in S_3} \pi, \quad e_2 = \frac{1}{6} \sum_{\pi \in S_3} \text{sign}(\pi) \pi\]

\[e_3 = \frac{1}{3} \left( \text{id} + \omega(123) + \omega^2(132) \right), \quad e_4 = \frac{1}{3} \left( \text{id} + \omega^2(123) + \omega(132) \right)\]

Check: \[e_i^2 = e_i, \quad e_i e_j = e_j e_i = 0 \text{ for } i \neq j, \quad e_i = \text{id}\]

\[e_i \mapsto E_{ii} \text{ under ring isomorphism}\]

Decomp. into simple modules:

\[L_1 = Re_1 = Ce_1, \quad L_2 = Re_2 = Ce_2\]

\[L_3 = Re_3 = \text{span}_\mathbb{C} \left\{ \text{id} + \omega(123) + \omega^2(132), (12) + \omega(23) + \omega^2(13) \right\}\]

\[L_4 = Re_4 = \text{span}_\mathbb{C} \left\{ \text{id} + \omega^2(123) + \omega(132), (12) + \omega(13) + \omega^2(23) \right\}\]

As left \(CS_3, \quad CS_2 \cong L_1 \oplus L_2 \oplus L_3 \oplus L_4 \]

\[L_3 \cong L_4\]

Over non-algebraically closed fields, numerology can be slightly different.
Ex: \( G = \langle \sigma : \sigma^4 = 1 \rangle \) \quad F = \mathbb{R} \quad x^4 - 1 = (x-1)(x+1)(x^2+1)

\[ \text{IRG} \cong \mathbb{R}[x]/\langle x^4 - 1 \rangle \cong \mathbb{R}[x]/\langle x-1 \rangle \otimes \mathbb{R}[x]/\langle x+1 \rangle \otimes \mathbb{R}[x]/\langle x^2+1 \rangle \]
\[ \cong \mathbb{R} \otimes \mathbb{R} \otimes \mathbb{C} \]
\( r = 3 \quad n_1 = n_2 = n_3 = 1 \quad \sum_{i=1}^{3} n_i^2 = 3 = |G| \quad \sum_{i=1}^{3} n_i^2 \text{ dim}_{\mathbb{R}}(\Delta_i) = 1^2 + 1^2 + 1^2 + 2 = |G| = 4 \)

Ex: \( G = Q_8 = \langle i, j \mid i^4 = j^4 = 1, \quad i^2 = j^2, \quad ij = -ji \rangle \)
\[ = \{ \pm 1, \pm i, \pm j, \pm k \} \quad \text{with} \quad i^2 = j^2 = k^2 = -1, \quad ij = -ji = k \quad jk = -kj = i \quad ki = -ik = j \]
\( \text{IRQ}_8 \cong ? \) what are irreducible reps of \( Q_8 \) over \( \mathbb{R} \)?
\( k i = -i k = j \)

A 4-dim' rep of \( Q_8 \) given by real Hamiltonian quaternions:
\[ \mathcal{H} = \left\{ a+i b + j c + k d : a, b, c, d \in \mathbb{R} \right\} \]
\[ \cong \text{IRQ}_8 \bigg/ \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bigg\rangle \]
\( \Delta = \text{group elt.} \)

In \( \text{IRQ}_8 \), \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) image in \( \mathcal{H} \) is zero.

Remark: \( \mathcal{H} \) is a division ring!

Idempotents in \( \mathcal{H} \)? \( e^2 = e, \quad e + 0 \quad \Rightarrow \quad e = 1 \)

only 1 idempotent \( e \) \( \text{mult. by } e^2 \)
\( \Rightarrow \mathcal{H} \text{ is an irreducible } Q_8 \)-module.

\( \text{dim}_{\mathbb{R}}(\mathcal{H}) = 4 \) \( \Rightarrow \mathcal{H} \text{ gives 4-dim' representation of } Q_8 \)
w.r.t. basis $i, j, k, k^2$ we can write down homomorphism $Q_8 \to \text{GL}_4(\mathbb{R})$

\[ i \cdot 1 = i, \quad i \cdot i = -1, \quad i \cdot j = k, \quad i \cdot k = -j \]

\[ i \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{etc.} \]

Other irreducible reps of $Q_8$ over $\mathbb{R}$:

1 dim. rep's:

\[ \Psi_1 (i) = 1 \quad \Psi_1 (j) = 1 \quad \text{(trivial rep)} \]

\[ \Psi_2 (i) = -1 \quad \Psi_2 (j) = 1 \]

\[ \Psi_3 (i) = 1 \quad \Psi_3 (j) = -1 \]

\[ \Psi_4 (i) = -1 \quad \Psi_4 (j) = -1 \]

\[
\begin{bmatrix} \mathbb{M}_1(\mathbb{R}) \times \mathbb{M}_1(\mathbb{R}) \times \mathbb{M}_1(\mathbb{R}) \times \mathbb{M}_1(\mathbb{R}) \times \mathbb{M}_1(\mathbb{H}) \end{bmatrix} \]

\[ r = 5 \quad n_1 = \ldots = n_5 = 1 \]

\[
\sum_{i=1}^{5} n_i^2 \text{dim}_{\mathbb{R}}(A_i) = 1^2 + 1^2 + 1^2 + 1^2 + 4 = 8 = |GL| \]

\[ \text{a noncomm. division ring} \]

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