Today: Start §18.2

Hwk 9 due Fri 4/10 (Moodle or email)

Bonus Office Hour: Fri 4/10, 11am-noon (my zoom "office")

Hwk 10 due Fri 4/17, posted online tomorrow

§18.2 WEDDERBURN'S THEOREM AND APPLICATIONS

Notation recall:

An R-module M is simple if its only submodules are 0 and M.

A division ring is a ring R with $1 \neq 0$ s.t. every $a \in R$ has a multiplicative inverse $b \in R$ $(ab = ba = 1)$.

(commutative division ring = field)

Lemma: If M, N are simple R-modules and $\varphi: M \to N$ is a non-zero R-mod. hom, then φ is an isomorphism.

(Proof) φ nonzero $\Rightarrow$ $\ker(\varphi) \subsetneq M$ $\Rightarrow$ $\ker(\varphi) = 0$.

$\Downarrow$

$\{0\} \subsetneq \text{image}(\varphi) \subseteq N$ $\Rightarrow$ $\text{image}(\varphi) = N$. $\square$

Wedderburn's Thm: Let R be a ring with $1 \neq 0$. TFAE:

1) Every R-module is projective.
2) Every R-module is injective.
3) Every R-module is completely reducible.
4) The ring R is isomorphic (as a left R-module) to a direct sum

$$R = L_1 \oplus L_2 \oplus \ldots \oplus L_n$$

where L_i is a simple R-module with $L_i = Re_i$ for some $e_1, \ldots, e_n$ satisfying

$$e_i e_j = 0 \text{ for } i \neq j, \quad e_i^2 = e_i, \quad \sum_{i=1}^{n} e_i = 1.$$

5) R is isomorphic (as a ring) to a direct product of matrix rings over division rings,

$$R \cong M_{n_1}(\Delta_1) \times \dots \times \underbrace{M_{n_r}(\Delta_r)}$$

ring of $n_r \times n_r$ matrices with entries in Δ_r

where $n_1, \dots, n_r \in \mathbb{Z}_+$, $\Delta_1, \dots, \Delta_r$ are division rings.

Moreover $r, n_1, \dots, n_r, \Delta_1, \dots, \Delta_r$ are unique determined (up to isom.) by R.

(Proof in DF Exercises 18.2 1-10.)

Def A ring R satisfying (1)-(5) is called <u>semisimple</u>.

Ex: $R = M_1(\mathbb{C}) \times M_2(\mathbb{C}) = \left\{ \left(\begin{array}{c|cc} a & 0 & 0 \\ \hline 0 & b & d \\ 0 & c & e \end{array} \right) : a,b,c,d,e \in \mathbb{C} \right\}$

$(r=2)$

$(n=3)$ let E_{ij} be matrix with 1 in (i,j) entry, 0's elsewhere.

Take $e_1 = E_{11}$, $e_2 = E_{22}$, $e_3 = E_{33}$.

Check: $E_{ii}E_{jj} = 0$ for $i \neq j$, $E_{ii}^2 = E_{ii}$, $E_{11} + E_{22} + E_{33} = I$

$RE_{11} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a \in \mathbb{C} \right\}$ $RE_{22} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & c & 0 \end{pmatrix} : b,c \in \mathbb{C} \right\}$ $RE_{33} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d \\ 0 & 0 & e \end{pmatrix} : d,e \in \mathbb{C} \right\}$

As a left R-module

$$R \cong RE_{11} \oplus RE_{22} \oplus RE_{33}.$$

Def: Nonzero $e \in R$ s.t. $e^2 = e$ is <u>idempotent</u>.

Two idempotents $e_1, e_2 \in R$ are <u>orthogonal</u> if $e_1 e_2 = e_2 e_1 = 0$.

Idempotent $e \in R$ is <u>primitive</u> if it is not the sum of two orthogonal idempotents.

Ex: $R = M_1(\mathbb{C}) \times M_2(\mathbb{C})$

E_{ii}, $E_{ii} + E_{jj}$, $I = E_{11} + E_{22} + E_{33}$ all idempotent

E_{ii} primitive, pairwise orthogonal

$\Rightarrow E_{ii} + E_{jj}$, I not primitive

Ex: FG where G finite group, $\mathrm{char}(F) \nmid |G|$.

Maschke's Thm: For any FG-module V and submodule $U \subseteq V$ then there exists submod. W with $V = U \oplus W$.

$\Rightarrow$ Every FG-module is injective.

$\Rightarrow$ FG is semisimple.

For $R = \mathbb{C}G$, all division rings Δ_j in decomp are just $\mathbb{C}$.

Prop: If Δ is a division ring and Δ is a finite dim'l vec. space over field F where $F^{alg} = F$ and $F \subseteq Z(\Delta)$ center then $F = \Delta$.

(Proof) Let $\alpha \in \Delta$. Since $F \subseteq Z(\Delta)$, $F(\alpha)$ is commutative $\Rightarrow$ field.

$F(\alpha) \subseteq \Delta \Rightarrow F(\alpha)$ is finite dim'l over F

$\Rightarrow \alpha$ is algebraic over F

$\Rightarrow \alpha \in F$. $\square$

Then Wedderburn's Thm (5) says that for any finite group G,

$$\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C}). \qquad \text{(as rings)}$$

for some $n_1, \dots, n_r \in \mathbb{Z}_+$.

Remark:
- $M_n(\mathbb{C})$ commutative $\Longleftrightarrow n = 1$
- $\mathbb{C}G$ is commutative $\Longleftrightarrow G$ is abelian $\Longleftrightarrow n_1 = \dots = n_r = 1$

Prop: Let $R = M_n(\mathbb{C})$.

1) $\dim_{\mathbb{C}} R = n^2$

2) $Z(R) = \{\alpha I : \alpha \in \mathbb{C}\}$ and $\dim_{\mathbb{C}}(Z(R)) = 1$

3) Every simple left R-module is isomorphic to $L_1 = R \cdot E_{11}$.

(1) and (2) Check!

(3) $L_1 = RE_{11}$ is simple because $RE_{11} = RA$ for any nonzero $A \in RE_{11}$.

$A \in RE_{11} \Rightarrow A = \begin{pmatrix} | & | & & | \\ v & 0 & \cdots & 0 \\ | & | & & | \end{pmatrix}$ for some $v \in \mathbb{C}^n$.

$A \neq 0 \Rightarrow v \neq 0 \Rightarrow \exists\, B \in M_n(\mathbb{C}) = R$ s.t. $Bv = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\Rightarrow BA = E_{11}$

$\Rightarrow E_{11} \in RA$.

If M is a simple, nonzero R-module, then take $m \in M$, $m \neq 0$.

$$m = I \cdot m = (E_{11} + E_{22} + \dots + E_{nn}) \cdot m = E_{11} m + E_{22} \cdot m + \dots + E_{nn} \cdot m$$

$\Rightarrow E_{ii} \cdot m \neq 0$ for some i.

Consider map $\varphi : RE_{ii} \to M$ given by $\varphi(rE_{ii}) = rE_{ii} \cdot m$.

Nonzero module hom of simple modules $\Rightarrow \varphi$ is an isomorphism. $\square$

Prop: Let $R = M_{n_1}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C})$.

1) $\dim_{\mathbb{C}}(R) = \sum_{i=1}^{r} n_i^2$

2) $Z(R) = Z(M_{n_1}(\mathbb{C})) \times \dots \times Z(M_{n_r}(\mathbb{C}))$ and $\dim_{\mathbb{C}}(Z(R)) = r$

3) The simple R-modules are the simple $M_{n_i}(\mathbb{C})$ modules on which other $M_{n_j}(\mathbb{C})$ act trivially.

RELATION TO CONJUGACY CLASSES OF G:

Let $K_1, \dots, K_s$ be the conjugacy classes of G.

Take

$$X_i = \sum_{g \in K_i} g \in \mathbb{C}G.$$

$K_1, \dots, K_s$ pairwise disjoint $\Rightarrow X_1, \dots, X_s \in \mathbb{C}G$ linearly indep. over $\mathbb{C}$.

For $h \in G$,

$$h^{-1} X_i h = \sum_{g \in K_i} h^{-1} g h = \sum_{\tilde{g} \in K_i} \tilde{g} = X_i$$

$\Rightarrow X_i h = h X_i \quad \forall h \in G$

$\Rightarrow X_i \in Z(\mathbb{C}G).$

<u>Claim</u>: $X_1, \ldots, X_s$ are a $\mathbb{C}$-basis for $Z(\mathbb{C}G)$.

(spanning) $Y \in \sum_{g \in G} \alpha_g \cdot g \in Z(\mathbb{C}G)$.

$$\Rightarrow h^{-1} Y h = \underbrace{\sum_{g \in G} \alpha_g h^{-1} g h}_{\text{coeff of } g \text{ is } \alpha_{hgh^{-1}}} = \underbrace{\sum_{g \in G} \alpha_g g}_{\text{coeff of } g \text{ is } \alpha_g} = Y \quad \forall h \in G.$$

$\Rightarrow \alpha_{hgh^{-1}} = \alpha_g \quad \forall h \in G.$

$\Rightarrow \alpha_{g_1} = \alpha_{g_2}$ whenever $g_1, g_2 \in K_i$

$\Rightarrow Y \in \text{span}_{\mathbb{C}} \{X_1, \ldots, X_s\}.$

$\square$

<u>Thm</u>: Let G be a finite group.

1) $\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$
2) $|G| = \sum_{i=1}^{r} n_i^2$
3) $r = \#$ conjugacy classes of $G = \dim_{\mathbb{C}}(Z(\mathbb{C}G))$
4) $\mathbb{C}G$ has r distinct isomorphism types of irreducible modules (with degrees $n_1, \ldots, n_r$).

Ex: $G = S_3$