Today: finish 18.1
Thurs: start 18.2
Hwk 9 due Fri

Concerns / Suggestions for qualifying exam logistics? Email me.

Recall: G = finite group
F = field
V = F-vectorspace

$\varphi : G \to GL(V)$ group hom.
= linear representation of G

$g \cdot v = \varphi(g)(v) \in V$

$\{\text{linear reps of } G \text{ over } F\} \longleftrightarrow \{\underline{FG}\text{-modules}\}$

group ring with coeff in F

Def: A subspace $U \subseteq V$ is *G-invariant* (or *G-stable*) if $g \cdot u \in U$ for all $g \in G$, $u \in U$.

Ex: $G = S_3$ $V = \mathbb{R}^3$ $\varphi : G \to GL_3(\mathbb{R})$ $\varphi(\pi) = P_\pi$ (perm matrix)

$U = \mathbb{R}\{(1,1,1)\}$ is G-invariant

$\pi \cdot (a_1, a_2, a_3) = (a_{\pi(1)}, a_{\pi(2)}, a_{\pi(3)})$

$W = \{a \in \mathbb{R}^3 : a_1 + a_2 + a_3 = 0\}$ is also G-invariant

$\Leftrightarrow a_{\pi(1)} + a_{\pi(2)} + a_{\pi(3)} = 0$

Remark: A subspace $U \subseteq V$ is G-invariant
$\Longleftrightarrow U$ is an FG-submodule of V

$(\sum \alpha_g \cdot g) \cdot u \in U$
for all $\sum \alpha_g g \in FG$
all $u \in U$

Def: Let R be a ring and M a nonzero R-module

1) M is *reducible* if it has a nontrivial submodule $0 \subsetneq N \subsetneq M$ and *irreducible* (or *simple*).

2) M is *decomposable* if it can be written as $M_1 \oplus M_2$ for nonzero submod. M_1, M_2, and *indecomposable* otherwise

3) M is completely reducible if it a direct sum of irreducible submodules

Def: A representation $G \to GL(V)$ is $(\ast)$ if the corresponding FG-module is $(\ast)$.

$(\ast)$ = reducible, decomposable, ...

Remark: irreducible $\Rightarrow$ indecomposable
$\not\Rightarrow$ completely reducible

(converses don't always hold)

Translation to matrices

Reducibility: $U \subseteq V$ an FG-submodule (i.e. G-inv. subspace)

Choose basis $u_1, \dots, u_m$ for U, extend to basis $u_1, \dots, u_m, v_{m+1}, \dots, v_n$ for V. W.r.t. this basis $\varphi(g)$ has the form

$$\begin{array}{cc} & \begin{array}{cc} m & \end{array} \\ \begin{array}{c} m \\ n-m \end{array} & \left[\begin{array}{c|c} A_g & C_g \\ \hline 0 & B_g \end{array}\right] \end{array}$$

$g \mapsto A_g$ gives matrix rep of $\varphi|_U$

$g \mapsto B_g$ gives matrix rep of φ on V/U.

Decomposability: $V = U \oplus W$ for U, W G-invariant

Choose basis $u_1, \dots, u_m$ for U and $w_1, \dots, w_{n-m}$ for W.

$\Rightarrow u_1, \dots, u_m, w_1, \dots, w_{n-m}$ are a basis for V.

W.r.t. to this basis $\varphi(g)$ has block diagonal form $\begin{array}{cc} & \begin{array}{cc} m & n-m \end{array} \\ \begin{array}{c} m \\ n-m \end{array} & \left(\begin{array}{c|c} A_g & 0 \\ 0 & B_g \end{array}\right) \end{array}$

$g \mapsto A_g$ given by $\varphi|_U$

$g \mapsto B_g$ given by $\varphi|_W$

Ex: $G = D_6$ $\quad \varphi(\sigma^k) = \begin{pmatrix} \cos(2\pi k/3) & -\sin(2\pi k/3) \\ \sin(2\pi k/3) & \cos(2\pi k/3) \end{pmatrix} \in GL_2(\mathbb{R})$

No nontrivial subspace of $\mathbb{R}^2$ fixed by $\varphi(\sigma)$ $\Rightarrow$ rep. is irreducible

Ex: $G = S_3$ $\quad \varphi(\pi) = P_\pi \in GL_3(\mathbb{R})$ (perm matrices)

reducible: nontrivial submodule $U = \mathbb{R}\{(1,1,1)\}$

decomposable: $\mathbb{R}^3 = U \oplus W$ where $W = \{a \in \mathbb{R}^3 : a_1 + a_2 + a_3 = 0\}$

also G-invariant.

U, W irreducible $\Rightarrow$ rep. is completely reducible

$\varphi|_W$ isom. to rep of D_6

Ex: $G = \langle \sigma : \sigma^p = 1 \rangle \cong \mathbb{Z}/p\mathbb{Z}$, p prime $\quad F = \mathbb{F}_p$, $V = \mathbb{F}_p^2$ with basis $\{e_1, e_2\}$

$\varphi : G \to GL_2(\mathbb{F}_p)$ $\quad \varphi(\sigma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \sigma \cdot e_1 = e_1$, $\sigma \cdot e_2 = e_1 + e_2$

$\varphi(\sigma^k) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

reducible: $\operatorname{span}\{e_1\}$ nontrivial G-invariant subspace ($\mathbb{F}_pG$-submod.)

indecomposable: If rep. were decomp. then $V = U \oplus W$ where U, W are G-inv. and $\dim(U) = \dim(W) = 1$.

$\Rightarrow \varphi(\sigma)$ is diagonal w.r.t. some basis of V. $\cancel{\#}$

not completely reducible

$\varphi(\sigma)$ is a Jordan block

Thm (Maschke's Thm)

includes $\operatorname{char}(F) = 0$

Let G be a finite group and F a field where $\operatorname{char}(F) \nmid |G|$. For any FG-module V and submodule $U \subseteq V$ there is an FG-submodule $W \subseteq V$ so that $V = U \oplus W$.

Cor: If $\operatorname{char}(F) \nmid |G|$ and $\varphi: G \to GL(V)$ is a linear rep. with V finite dim'l, then φ is completely reducible.

(Proof) By Maschke's Thm, reducible $\Rightarrow$ decomposable.

Induct on $\dim(V)$. If $\dim(V)=1$, V irreducible $\Rightarrow$ done.

If $\dim(V)=n>1$, either V is irreducible ($\Rightarrow$ done) or $V = U \oplus W$ where $\dim(U) < \dim(V)$ and $\dim(W) < \dim(V)$.

$\Rightarrow U, W$ completely reducible $\Rightarrow V$ completely reducible.

(Proof of Maschke's Thm) Take $U \subseteq V$ a FG-submodule.

Let $\pi: V \to U$ be an F-linear map s.t. $\pi(u)=u$ for all $u \in U$.

Define $\psi: V \to U$ by

$$\psi(v) = \underbrace{\frac{1}{|G|}}_{|G| \text{ invertible in } F} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot v)$$

Note: $\pi(g^{-1}\cdot v) \in U \Rightarrow g \cdot \pi(g^{-1} \cdot v) \in U \Rightarrow \psi(v) \in U$.
(by G-invariance of U)

Claim: (1) $\psi(u) = u$ for all $u \in U$

(2) ψ is an FG-module homomorphism

(1) Take $u \in U$. $\psi(u) = \frac{1}{|G|} \sum_{g \in G} g \cdot \underbrace{\boxed{\pi(g^{-1}\cdot u)}}_{= g^{-1} \cdot u \text{ since } g^{-1}\cdot u \in U}$

$= \frac{1}{|G|} \sum_{g \in G} g \cdot g^{-1} \cdot u$

$= \frac{1}{|G|} |G| \, u = u$ $\square_{\text{Claim 1}}$

(2) π is F-linear $\Rightarrow$ Ψ is F-linear (Check!)

Need is $\Psi(h\cdot v) = h\cdot\Psi(v)$ $\forall v\in V$ $\forall h\in G$.

$$h^{-1}\cdot\Psi(h\cdot v) = \frac{1}{|G|}\sum_{g\in G} h^{-1}\cdot g\cdot\pi(g^{-1}\cdot h\cdot v)$$

$g \mapsto h^{-1}\cdot g = \tilde{g}$ gives bijection $G\to G$

$$= \frac{1}{|G|}\sum_{\tilde{g}\in G} \tilde{g}\cdot\pi(\tilde{g}^{-1}\cdot v)$$

$$= \Psi(v)$$

$\square_{\text{Claim 2}}$

Let $W = \ker(\Psi)$. Then W is a FG-submodule of V.

Moreover the short exact sequence

$$0 \to W \xrightarrow{i_W} V \xrightarrow{\Psi} U \to 0$$

with $i_U : U \to V$,

splits, since $i_U \circ \Psi = id_U$.

$\Rightarrow V \cong U\oplus W$ as FG-modules.

$\square_{\text{Thm}}$

If $\mathrm{char}(F)\nmid |G|$, we can study all representations by studying direct sums of irreducible representations.