Today: Intro to Representation Theory (DF 18.1) (PART I)

F = a field $GL(V) = \{\text{invertible linear transf. } V \to V\}$

V = a vectorspace over F $GL_n(F) = \{n \times n \text{ invertible matrices over } F\}$

G = a finite group

Def: A linear respresentation of G is a group homomorphism

$$\varphi : G \to GL(V)$$

(gives action of G on V by $g \cdot v = \varphi(g)(v)$)

The degree of this representation is $\dim_F(V)$.

A matrix representation of G is a group hom

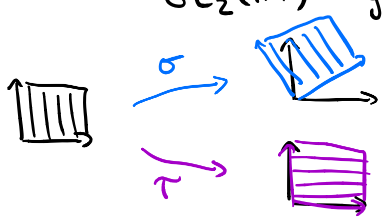
$$\varphi : G \to GL_n(F).$$

Representation is faithful if φ is injective.

Ex (trivial representation): $V = F^1$ $\varphi : G \to GL(V)$, $g \mapsto \mathrm{id}_V$

Ex: $G = D_{2n} = \langle \sigma, \tau : \sigma^n = 1, \tau^2 = 1, \sigma\tau = \tau\sigma^{-1} \rangle$

$\varphi : G \to GL_2(\mathbb{R})$ given by $\varphi(\sigma) = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$



$$\varphi(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Ex: $G = S_n$ $V = F^n$ with basis $e_1, \dots, e_n$

$\varphi : G \to GL_n(F)$

$\varphi(\pi) = P_\pi$ where $P_\pi e_i = e_{\pi(i)}$

$(n=3)$

$$\mathrm{id} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$(23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (132) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Ex: $G = \mathbb{Z}$ $\qquad \varphi : G \to GL_2(\mathbb{R})$

$$\varphi(1) = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$$

Not a faithful representation

$\varphi(a) = \varphi(b)$ whenever $a \equiv b \mod n$

$\Rightarrow \varphi$ not injective

$\Rightarrow$ rep. not faithful.

Connection with modules over group ring (PART II)

linear transf. over F $\longleftrightarrow$ $F[x]$-modules

rep. of G over F $\longleftrightarrow$ FG-modules

Group ring over F:

$$FG = \left\{ \sum_{g \in G} \alpha_g g \ : \ \alpha_g \in F \right\}$$

$\leftarrow$ formal F-linear comb of elts of G

addition componentwise

multiplication given by mult. in G and F-linearity

Remark: $FG \cong F^{|G|}$ as F-vectorspace

Ex: $G = S_3 \quad F = \mathbb{R}$

$$(\mathrm{id} + 5(12)) \cdot (2(13) + 7(132))$$

$$= 2(13) + 7(132) + 10 \cdot \underbrace{(12)(13)}_{(132)} + 35\,\underbrace{(12)(132)}_{(13)}$$

$$= 37(13) + 17(132)$$

Remark: FG is a commutative ring $\iff G$ is abelian

$F \in \mathrm{center}(FG) \Rightarrow FG$ is an F-algebra

A linear representation $\varphi: G \to GL(V)$ turns V into an FG-module:

$$\left(\sum_{g \in G} \alpha_g g\right) \cdot v = \sum_{g \in G} \alpha_g \varphi(g) v$$

Check: this satisfies module axioms!

Conversely, suppose that V is an FG-module.

Then $F \cdot \mathrm{id}_G$ gives an action of F on V

$\Rightarrow V$ is an F-module $=$ F-vectorspace

Moreover for any $g \in G$, $v, w \in V$, $\alpha, \beta \in F$

$$\left.\begin{aligned} g \cdot (\alpha v + \beta w) &= g \cdot \alpha v + g \beta w \\ &= \alpha g v + \beta g w \end{aligned}\right\}$$

g acts on V by a linear transformation $\varphi(g): V \to V$

Check: $\varphi: G \to GL(V)$ is a group hom.

Ex: $G = S_3 \quad F = \mathbb{R} \quad \varphi(\pi) = P_\pi \in GL_3(\mathbb{R})$

makes $\mathbb{R}^3$ into a $\mathbb{R}S_3$-module

$$(5(12) + 3(13))(a_1, a_2, a_3) = 5(a_2, a_1, a_3) + 3(a_3, a_2, a_1)$$

Def Two representations of G are <u>equivalent</u> (PART II)
if corresponding FG-modules are isomorphic,
<u>inequivalent</u> otherwise.

Prop: Reps. $\varphi: G \to GL(V)$, $\psi: G \to GL(W)$ are equivalent
$\iff$ there is some invertible $T: V \to W$ (F-linear transf)
s.t.

$$T \circ \varphi(g) \circ T^{-1} = \psi(g) \qquad \forall g \in G.$$

(Proof) ($\Rightarrow$) Take $T: V \to W$ to be isom. as FG-modules.

$F \cdot \mathrm{id} \subseteq FG \quad \Rightarrow \quad T$ is an F-mod hom (i.e. F-linear transf.)

$$T(g \cdot v) = g \cdot T(v) \qquad \text{(by } FG\text{-mod hom)}$$

$$T \circ \varphi(g)(v) = T(\varphi(g)(v)) \qquad \psi(g)(T(v)) = \psi(g) \circ T(v)$$

$$\Rightarrow T \circ \varphi(g) = \psi(g) \circ T$$

($\Leftarrow$) Similar. $\square$

Remark: Equivalent reps. have same deg.

Ex: $G = S_3$ $\quad \varphi: S_3 \to GL_3(\mathbb{R}) \quad \varphi(g) = I_3 \quad \forall g \in G$
$\quad \psi: S_3 \to GL_3(\mathbb{R}) \quad \psi(g) = P_g$ (perm. matrix)

$T \in GL_3(\mathbb{R}) \qquad T \circ \varphi(g) \circ T^{-1} = I_3 \neq \psi(g)$ when $g \neq id$

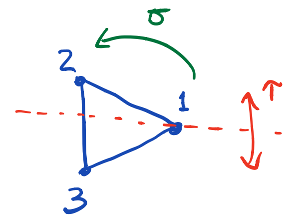
$\Rightarrow$ inequiv. reps.

Given reps. $\varphi: G \to GL(V)$, $\psi: G \to GL(W)$,
direct sum of V, W as FG-modules gives representation
$\varphi \oplus \psi: G \to GL(V \oplus W)$.

$$g\cdot(v,w) = (g\cdot v,\ g\cdot w) = (\varphi(g)\cdot v,\ \psi(g)w)$$

If $A(g)$, $B(g)$ are matrices rep. $\varphi(g)$, $\psi(g)$ (resp.) w.r.t. some bases of V, W, then $(\varphi\oplus\psi)(g)$ is represented by

$$\left(\begin{array}{c|c} A(g) & 0 \\ \hline 0 & B(g) \end{array}\right)$$



Ex: $G = S_3 \cong D_6$ $\quad (123) \longleftrightarrow \sigma \quad (23) \longleftrightarrow \tau$

$\varphi : D_6 \to GL_2(\mathbb{R})$

$\psi : S_3 \to GL_3(\mathbb{R})$ (perm)

inequiv (different deg)

$$\psi(g)\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad \forall g\in S_3$$ $\Rightarrow$ ψ restricts to trivial rep of G on $\mathbb{R}\{(1,1,1)\}$

In fact, ψ is equivalent to $1\oplus\varphi$ (where 1 is trivial rep)

For all $g\in G$

$$T\psi(g)T^{-1} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\varphi(g)$}} \\ 0 & & \end{array}\right) \quad \forall g \quad \text{where } T = \cancel{\begin{pmatrix} 1 & 1 & 1 \\ 1 & \cos(2\pi/3) & -\sin(2\pi/3) \\ 0 & \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}}$$

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$$T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \cos(2\pi/3) & \cos(2\pi/3) \\ 0 & \sin(2\pi/3) & -\sin(2\pi/3) \end{pmatrix}$$