Today: finish 17.2

Hwk 8 due tomorrow (Moodle)

Hwk 9 due Fri 4/10 (posted tomorrow)

Sat = "make-up Tues"

I will post lectures on 18.1. Watch them before Tues. 4/7

From last time:

$G =$ a group

$A =$ a G-module $=$ an abelian group with homomorphism $G \to \mathrm{Aut}(A)$

$= (\mathbb{Z}G)$-module

group ring of formal $\mathbb{Z}$-linear comb. of elt. of G

$A^G = \{a : ga = a \;\forall g \in G\}$

$= H^0(G, A) \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$ (with trivial action of G on $\mathbb{Z}$)

More generally: $H^n(G,A) \cong \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$

(Explicit connection comes from "bar resolution" of $\mathbb{Z}$ as $(\mathbb{Z}G)$-module DF p799)

Ex: $G =$ cyclic group of order $m = \langle \sigma : \sigma^m = 1 \rangle$

$\mathbb{Z}G = \{ a_0 \cdot 1 + a_1\sigma^1 + a_2\sigma^2 + \dots + a_{m-1}\sigma^{m-1} : a_0, a_1, \dots, a_{m-1} \in \mathbb{Z}\}$

Define $\mathrm{aug}: \mathbb{Z}G \to \mathbb{Z}$ by $\mathrm{aug}\left(\sum_{i=0}^{m-1} a_i\sigma^i\right) = \sum_{i=0}^{m-1} a_i$

This is a G-module hom.

take $a_{-1} = a_{m-1}$

$\mathrm{aug}\left(\sigma \cdot \sum_{i=0}^{m-1} a_i\sigma^i\right) = \mathrm{aug}\left(\sum_{i=0}^{m-1} a_i\sigma^{i+1}\right) = \mathrm{aug}\left(\sum_{i=0}^{m-1} a_{i-1}\sigma^i\right)$

$$= \sum_{i=0}^{m-1} a_i = \sigma \cdot \sum_{i=0}^{m-1} a_i$$

The kernel of aug is $(\sigma-1)\mathbb{Z}G$ (Check!)

Start to build a free res. of $\mathbb{Z}$ as a $\mathbb{Z}G$-mod:

$$\mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{aug} \mathbb{Z} \to 0$$

The kernel of mult. by $\sigma-1$

$$= \left\{ \sum_{i=0}^{m-1} a_i\sigma^i : (\sigma-1)\sum_{i=0}^{m-1} a_i\sigma^i = 0 \right\}$$

$$= \left\{ \sum_{i=0}^{m-1} a_i\sigma^i : a_0 = a_1 = \dots = a_{m-1} \right\} = N\cdot\mathbb{Z}G$$

where $N = 1+\sigma+\sigma^2+\dots+\sigma^{m-1}$.

The kernel of mult. by N

$$0 = \left(\sum_{j=0}^{m-1}\sigma^j\right)\left(\sum_{i=0}^{m-1} a_i\sigma^i\right) = \sum_{i,j} a_i\sigma^{i+j} = \sum_{i=0}^{m-1} a_i \cdot \left(\sum_{j=0}^{m-1}\sigma^j\right)$$

$$= aug\left(\sum a_i\sigma^i\right)\cdot N$$

So ker of mult. by N

$= \ker$ of aug $= (\sigma-1)\mathbb{Z}G$

Fix i. $\quad a_i\sigma^i \cdot \sum_{j=0}^{m-1}\sigma^j = a_i \boxed{\sum_{j=0}^{m-1}\sigma^{i+j}} = \boxed{\sum_{k=0}^{m-1}\sigma^k}$

$$\{0,1,\dots,m-1 \bmod m\} = \{i, i+1, \dots, i+m-1 \bmod m\}$$

$m=3 \quad i=1$

$$\sigma\cdot(1+\sigma+\sigma^2) = \sigma+\sigma^2+\sigma^3 = \sigma+\sigma^2+1$$

Free resol. of $\mathbb{Z}$ as a $\mathbb{Z}G$-module:

$$\dots \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{aug} \mathbb{Z} \to 0$$

To find $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$, take $\mathrm{Hom}_{\mathbb{Z}G}(-, A)$

(and drop "$\mathbb{Z}$" term) to get cochain complex:

$$0 \to \underset{\underset{A}{\cong}}{\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A)} \xrightarrow{\sigma-1} \underset{\underset{A}{\cong}}{\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A)} \xrightarrow{N} \underset{\underset{A}{\cong}}{\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A)} \xrightarrow{\sigma-1} \cdots$$

$$\mathrm{Ext}^0_{\mathbb{Z}G}(G,A) = \ker(\sigma-1) \cong \{a \in A : (\sigma-1)\cdot a = 0\} = A^G$$
$$= H^0(G,A).$$

$$\underset{\underset{H^n(G,A)}{\parallel}}{\mathrm{Ext}^n_{\mathbb{Z}G}(G,A)} = \begin{cases} \ker(\sigma-1)/NA & \text{if } n \text{ even, } n \geq 2 \\ \ker(N)/(\sigma-1)A & \text{if } n \text{ odd} \end{cases}$$

Ex: $m = 2d$ $\quad G = \langle \sigma : \sigma^{2d} = 1 \rangle \quad A = \mathbb{Z}/6\mathbb{Z}$

Define action of G on A by $G \to \mathrm{Aut}(A)$

$$\mathrm{Aut}(A) = \langle \varphi : \varphi^2 = id \rangle \cong \mathbb{Z}/2\mathbb{Z}$$
$$\varphi(a) = -a \mod 6$$

$$\sigma^k \to \begin{cases} id & \text{if } k \text{ even} \\ \varphi & \text{if } k \text{ odd} \end{cases}$$

$$\sigma^k \cdot a = \begin{cases} a & \text{if } k \text{ even} \\ -a & \text{if } k \text{ odd} \end{cases}$$

$$\ker(\sigma-1) = A^G = 3\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \quad \to \ker(\sigma-1)/NA \cong \mathbb{Z}/2\mathbb{Z}$$

$$NA = \{(1+\sigma+\sigma^2+\dots+\sigma^{2d-1})\cdot a : a \in A\}$$
$$= \{a + \sigma\cdot a + \sigma^2 a + \dots + \sigma^{2d-1}\cdot a : a \in A\}$$
$$= \{a - a + a - \dots - a : a \in A\} = \{0\}$$

$$\ker(N) = A = \mathbb{Z}/6\mathbb{Z} \quad \to \ker(N)/(\sigma-1)A = (\mathbb{Z}/6\mathbb{Z})/(2\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

$$(\sigma-1)A = \{(\sigma-1)\cdot a : a \in A\}$$
$$= \{\sigma\cdot a - a : a \in A\} = \{-2a : a \in A\} = 2\mathbb{Z}/6\mathbb{Z}$$

$\Rightarrow H^n(G,A) \cong \mathbb{Z}/2\mathbb{Z}$ for all n

Wiki: de Rham cohomology

Def Let H be a subgroup of G and let A be an H-module.
The induced G-module is $M_H^G(A)$ is

$$M_H^G(A) = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)$$

$$\cong \{\text{Functions } f: G \to A : f(hg) = hf(g) \ \forall h \in H \text{ and } g \in G\}$$

The action of G on $M_H^G(A)$ is defined by $(g \cdot f)(x) = f(xg) \quad \forall x, g \in G$

Ex: $H = \langle \sigma : \sigma^{2d} = 1 \rangle \leq G = D_{4d} = \left\langle \sigma, \tau : \begin{array}{c} \sigma^{2d} = 1, \\ \sigma\tau = \tau\sigma^{-1} \end{array} \right\rangle$

$A = \mathbb{Z}/6\mathbb{Z} \qquad \sigma^k \cdot a = \begin{cases} a & \text{if } k \text{ even} \\ -a & \text{if } k \text{ odd} \end{cases}$

Then $\mathbb{Z}G = \left\{ \sum_{i=0}^{2d-1} a_i \sigma^i + \sum_{j=0}^{2d-1} b_j \sigma^j \tau \ : a_i, b_j \in \mathbb{Z} \right\}$

as a $\mathbb{Z}H$-module $\cong \mathbb{Z}H \oplus \mathbb{Z}H$

A $(\mathbb{Z}H)$-mod. hom $f: \mathbb{Z}G \to A$ is uniquely det. by values on G, and satisfy $f(\sigma g) = \sigma f(g)$.
$\Rightarrow f(\sigma^k) = \sigma^k \cdot f(1)$ and $f(\sigma^k \tau) = \sigma^k \cdot f(\tau)$.

Check G-action on $M_H^G(A)$.

$$\left(M_H^G(A)\right)^G \cong \left\{ f: G \to A \ : \ \begin{array}{ll} f(hx) = h f(x) & \forall h \in H \\ & \forall x \in G \\ \boxed{f(xg) = f(x)} & \forall g \in G \end{array} \right\}$$

0th cohom. grp $H^0(G, M_H^G(A))$

$\hookrightarrow f(g) = f(1) \Rightarrow f: G \to A$ constant

$$\cong \left\{ f: G \to A \ : \ \begin{array}{ll} f(1) = h \cdot f(1) & \forall h \in H \\ f(g) = f(1) & \forall g \in G \end{array} \right\}$$

$$= \{ f: G \to A \ : \ \exists\, a \in A^H \text{ s.t. } f(g) = a \ \forall g \in G \}$$

$$\cong A^H$$

0th cohom. group $H^0(H, A)$

More generally

<u>Shapiro's lemma</u>: $H^n(G, M_H^G(A)) \cong H^n(H, A)$.