MA 721 Today: finish 17.1, start 17.2
Thurs: finish 17.2, examples
Fri: Hwk 8 due
Sat: "makeup day"
I will post some intro lectures on representation theory (DF 18.1)

$X \otimes_R - \rightsquigarrow \mathrm{Tor}$

A = an R-module

Given a proj. resolution of A,

$$\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0$$

tensoring with X gives

$$\to X \otimes_R P_n \xrightarrow{1 \otimes d_n} X \otimes_R P_{n-1} \to \cdots \to X \otimes_R P_1 \xrightarrow{1 \otimes d_1} X \otimes_R P_0 \xrightarrow{1 \otimes d_0} X \otimes_R A \to 0$$

Exactness of proj. resolution of A $\Rightarrow$ $\mathrm{image}(1 \otimes d_{n+1}) \subseteq \ker(1 \otimes d_n)$
$\Rightarrow$ chain complex!

Def: For $n \geq 1$, define $\mathrm{Tor}_n^R(X, A) = \ker(1 \otimes d_n) / \mathrm{image}(1 \otimes d_{n+1})$

$\mathrm{Tor}_0^R(X, A) = X \otimes_R P_0 / \mathrm{image}(1 \otimes d_1)$

"n^{th} homology group derived from $X \otimes_R -$"

Chain complex $\left\{ \begin{array}{l} \to C_2 \to C_1 \to C_0 \to 0 \\ \to \text{homology groups } H_n(C^\bullet) \end{array} \right.$

cochain complex $\left\{ \begin{array}{l} 0 \to C^0 \to C^1 \to C^2 \to \cdots \\ \to \text{cohom. groups } H^n(C^\bullet) \end{array} \right.$

Prop: $\mathrm{Tor}_n^R(X, A)$ is independent of choice of projective resolution of A.

Thm: If $0 \to A \to B \to C \to 0$ is a short exact seq. of R-mod. there is a long exact sequence:

$$\begin{array}{l} \cdots \to \operatorname{Tor}_2^R(X,C) \\ \to \operatorname{Tor}_1^R(X,A) \to \operatorname{Tor}_1^R(X,B) \to \operatorname{Tor}_1^R(X,C) \\ \to X \otimes_R A \to X \otimes_R B \to X \otimes_R C \to 0 \end{array}$$

Ex: $R = \mathbb{Z}$ $X = \mathbb{Z}^m$ (free $\Rightarrow$ proj. $\Rightarrow$ flat) A = any $\mathbb{Z}$-mod.

$$\to X \otimes P_n \to \cdots \to X \otimes P_0 \to X \otimes A \to 0$$

($\mathbb{Z} \to R$ any ring)

is exact and $\operatorname{Tor}_n^{\mathbb{Z}}(X, A) = 0$ for $n \geq 1$.

Ex: $\operatorname{Tor}_n^R(A, B_1 \oplus B_2) = \operatorname{Tor}_n^R(A, B_1) \oplus \operatorname{Tor}_n^R(A, B_2)$ (Check!)

Cor: If A, B are finitely generated $\mathbb{Z}$-modules then $\operatorname{Tor}_n^R(A,B) = \operatorname{Tor}_n^R(t(A), t(B))$ where $t(A), t(B)$ are the torsion submod. of A, B (resp.).

$\mathbb{Z} \to$ PID ($A \cong R^m \oplus t(A)$ etc.)

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§17.2 Cohomology of groups

Let G be a group.

Def: A G-module is an abelian group, A, together with an action of G on A by automorphisms (i.e. group hom. $\varphi : G \to \operatorname{Aut}(A)$)

Write $ga = \varphi(g)(a) \in A$.

Recall the group ring (with coeff in $\mathbb{Z}$), $\mathbb{Z}G$, is the ring

$$\mathbb{Z}G = \Big\{ \underbrace{\sum_{g \in G} \alpha_g \, g}_{\text{Formal } \mathbb{Z}\text{-linear combinations.}} : \alpha_g \in \mathbb{Z},\ \{g : \alpha_g \neq 0\} \text{ finite} \Big\}$$

Additively looks like free $\mathbb{Z}$-mod. over G

Mult. defined by mult. in G and $\mathbb{Z}$-linearity.

Remark: G-modules are exactly ($\mathbb{Z}G$)-modules (in usual sense of a module over a ring)

Def: $A^G = \{a \in A : ga = a \;\; \forall g \in G\}$

denotes the set of elements fixed by all $g \in G$.

Ex: $A \trianglelefteq G$, A normal subgroup of G $\quad \varphi(g)(a) = gag^{-1}$

is group. hom $G \to \mathrm{Aut}(A)$

$\Rightarrow A$ is a G-module

$A^G = A \cap \text{center}(G)$

As a $\mathbb{Z}G$-module, $(g_1 + 2g_2)\cdot a = (g_1 a g_1^{-1})(g_2 a g_2^{-1})(g_2 a g_2^{-1})$

Ex: K/F Galois field extension, $G = \mathrm{Gal}(K/F)$

Take $A = (K, +)$ or $A = (K^*, \times)$ give G-modules

$A^G = F \qquad\qquad A^G = F^*$

Q: Given a short exact sequence of G-modules $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$

what can we say about A^G, B^G, C^G?

Note: for $a \in A^G$, $\varphi(a) = \varphi(ga) = g\varphi(a) \quad \Rightarrow \varphi(a) \in B^G$

$\Rightarrow \varphi(A^G) \subseteq B^G$

Similarly $\psi(B^G) \subseteq C^G$

$\to$ we get $\quad 0 \to A^G \xrightarrow{\varphi} B^G \xrightarrow{\psi} C^G$ is exact

but ψ can fail to be surjective.

Ex: $G = \mathbb{Z}/2\mathbb{Z} \quad A = B = \mathbb{Z} \quad C = \mathbb{Z}/2\mathbb{Z}$

with $g \cdot n = \begin{cases} n & \text{if } g = 0 \\ -n & \text{if } g = 1 \end{cases}$

$$\begin{array}{ccccccccc} 0 \to & A & \xrightarrow{\times 2} & B & \xrightarrow{\pi} & C & \to 0 \\ & \| & & \| & & \| & \\ & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & \end{array}$$

short exact sequence of G-mod.

but $A^G = \{0\} = B^G \qquad C^G = C = \mathbb{Z}/2\mathbb{Z}$

$\Rightarrow$ $0 \to A^G \to B^G \to C^G \to 0$ is not exact at C^G

Lemma: For any G-module A, $A^G \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$ (as groups)

(Here $\mathbb{Z}$ is a $\mathbb{Z}G$-module with trivial action of G, $g\cdot z = z$)

(Proof Sketch) For $\varphi \in \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$, let $a = \varphi(1)$.

For $g \in G$, $a = \varphi(1) = \varphi(\underbrace{g\cdot 1}_{\text{trivial action of } G \text{ on } \mathbb{Z}}) = g\cdot\varphi(1) = ga \implies a \in A^G$

Similarly for $a \in A^G$, there is a unique $\varphi \in \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$ with $\varphi(1) = a$.

Moreover $\varphi(1) \longleftrightarrow \varphi$ is a group hom. (Check!) $\square$

To study non-exactness of $-^G$, we could take proj. resolution of $\mathbb{Z}$ (as a $\mathbb{Z}G$-module) and look at $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$.

A concrete version of $\uparrow$:

Def: Let G be a finite group and let A be a G-module.

Define $C^0(G, A) = A$ and for $n \geq 1$

$C^n(G, A) = \{\text{functions } f : G^n \to A\}$ ← "n-cochains of G with coeff in A"

Remark: For all n, $C^n(G, A)$ is a abelian group.

$$(f_1 + f_2)(g_1, \dots, g_n) = f_1(g_1, \dots, g_n) \underset{\uparrow}{+} f_2(g_1, \dots, g_n)$$

group oper. in A

Def: The n^{th} <u>coboundary homomorphism</u> is a map

$d_n : C^n(G, A) \to C^{n+1}(G, A)$. For $f \in C^n(G, A)$,

$$\begin{aligned} d_n(f)(g_1, \dots, g_{n+1}) = {} & g_1 \cdot f(g_2, \dots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, \dots, g_n) \end{aligned}$$

Check: d_n is a group hom and $d_n \circ d_{n-1} = 0$ for all $n \geq 1$.

$$\Rightarrow \quad 0 \to A \xrightarrow{d_0} C^1(G,A) \xrightarrow{d_1} C^2(G,A) \to \cdots \xrightarrow{d_{n-1}} C^n(G,A) \xrightarrow{d_n} \cdots$$

is a cochain complex

Def: 1) $Z^n(G,A) = \ker(d_n)$ "n-cocycles"

2) $B^n(G,A) = \text{image}(d_{n-1})$ for $n \geq 1$ "n-coboundaries"

$B^0(G,A) = 1$

3) $H^n(G,A) = Z^n(G,A)/B^n(G,A) = \ker(d_n)/\text{image}(d_{n-1})$

"n^{th} cohomology group of G with coeff in A"

Ex: $(n=0)$ $\quad d_0 : \underset{A}{\underset{\parallel}{C^0(G,A)}} \to C^1(G,A)$

$a \in A$, $g \in G$ $\quad d_0(a)(g) = ga - a$

$$Z^0(G,A) = \ker(d_0) = \{a \in A : d_0(a)(g) = 0 \ \forall g \in G\} = A^G$$

$$H^0(G,A) = A^G/1 = A^G$$

Thm: If $0 \to A \to B \to C \to 0$ is short exact seq. of G-mod., then there is a long exact sequence of abel. groups

$$0 \to A^G \to B^G \to C^G \to H^1(G,A) \to H^1(G,B) \to H^1(C,G) \to H^2(G,A) \to \cdots$$