MA 721 Announcements:

Today: § 17.1

Hwk 8 due Fri 4/3 on Moodle (posted tomorrow)

Homological Algebra Recall

Chain complex = sequence of abelian groups (C_n) with maps $d_n : C_n \to C_{n-1}$ s.t. $\mathrm{im}(d_{n+1}) \subseteq \ker(d_n)$

$$\cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} \cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \to 0$$

Homology groups: $\ker(d_n)/\mathrm{im}(d_{n+1})$

Cochain complex = sequence of abelian groups $(C^n)_n$ with maps $d_n : C^{n-1} \to C^n$ s.t. $\mathrm{im}(d_n) \subseteq \ker(d_{n+1})$

$$0 \to C^0 \xrightarrow{d_1} C^1 \xrightarrow{d_2} C^2 \xrightarrow{d_3} \cdots \xrightarrow{d_n} C^n \xrightarrow{d_{n+1}} \cdots$$

Cohomology groups: $\ker(d_{n+1})/\mathrm{im}(d_n)$

Important examples of the day: Ext, Tor

induced from 1) projective resolution

2) functor $\mathrm{Hom}(-,X)$, $\mathrm{Hom}(X,-)$, $X \otimes_R -$

Def: Let A be an R-module. A projective resolution of A is an exact sequence

$$\cdots \to P_n \xrightarrow{\varphi_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} A \to 0$$

where P_i is a projective R-module for all i

(called "free resolution" if P_i free $\forall i$)

Remark: Every R-module A has a free ($\Rightarrow$ proj.) resolution

$P_0 =$ free module on a set of generators $\{x_i\}_{i \in I}$ for A

$\varphi_0 : P_0 \to A \qquad \varphi_0(\sum r_i x_i) = \sum r_i x_i \in A$

P_1 = free mod. on a generating set for $\ker(\varphi_0)$

$\varphi_1 : P_1 \to P_0$ with $\mathrm{im}(\varphi_1) = \ker(\varphi_0)$

P_i = free mod on a gen. set for $\ker(\varphi_{i-1})$

$\varphi_i : P_i \to P_{i-1}$ with $\mathrm{im}(\varphi_i) = \ker(\varphi_{i-1})$

Ex: $R = \mathbb{Z} \quad A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$

Gen set 1: $\{(1,0), (0,1)\}$

$P_0 = \mathbb{Z} \oplus \mathbb{Z} \qquad \varphi_0(a,b) = (a+2\mathbb{Z}, b+3\mathbb{Z}) \quad \underbrace{\ker(\varphi_0) = 2\mathbb{Z} \oplus 3\mathbb{Z}}_{\text{gen by } \{(2,0),(0,3)\}}$

$P_1 = \mathbb{Z} \oplus \mathbb{Z} \qquad \varphi_1(a,b) = a(2,0) + b(0,3) = (2a, 3b) \quad \ker(\varphi_1) = 0$

$P_i = 0$ for all $i \geq 2$

Proj. resolution $\cdots \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\varphi_1} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\varphi_0} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \to 0$

Gen set 2: $\{(1,1)\}$

$P_0' = \mathbb{Z} \qquad \varphi_0'(a) = (a+2\mathbb{Z}, a+3\mathbb{Z}) \quad \underbrace{\ker(\varphi_0') = 6\mathbb{Z}}_{\text{gen by } \{6\}}$

$P_1' = \mathbb{Z} \qquad \varphi_1'(a) = 6a \qquad \ker(\varphi_1') = 0$

$P_i' = 0$ for $i \geq 2$

Proj resolution': $\cdots \to 0 \to \mathbb{Z} \xrightarrow{\varphi_1'} \mathbb{Z} \xrightarrow{\varphi_0'} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \to 0$

$\boxed{\mathrm{Hom}_R(-, X)}$

For any R-module X, map $P_n \xrightarrow{\varphi_n} P_{n-1}$ induces a group hom

$\mathrm{Hom}_R(P_{n-1}, X) \xrightarrow{d_n} \mathrm{Hom}_R(P_n, X)$

$d_n(f) = f \circ \varphi_n$

$$\begin{tikzcd} P_n \arrow[r, "\varphi_n"] \arrow[dr, "d_n(f) = f \circ \varphi_n"'] & P_{n-1} \arrow[d, "f"] \\ & X \end{tikzcd}$$

Proj. resolution of A induces

$0 \to \mathrm{Hom}_R(A,X) \to \mathrm{Hom}_R(P_0, X) \xrightarrow{d_1} \mathrm{Hom}_R(P_1, X) \xrightarrow{d_2} \cdots \xrightarrow{d_n} \mathrm{Hom}_R(P_n, X) \xrightarrow{d_{n+1}} \cdots$

Exactness of proj resolution $\Rightarrow$ this is a cochain complex

(Check!) $\qquad$ $(\mathrm{im}(d_n) \subseteq \ker(d_{n+1}))$

Def: For $n \geq 1$, $\mathrm{Ext}^n_R(A,X) = \ker(d_{n+1})/\mathrm{im}(d_n)$

$n=0$ $\quad \mathrm{Ext}^0_R(A,X) = \ker(d_1)$

"n^{th} cohomology group of A derived from $\mathrm{Hom}_R(-,X)$"

Thm: For all n, $\mathrm{Ext}^n_R(A,X)$ is independent of the choice of projective resolution of A.

Prop: $\mathrm{Ext}^0_R(A,X) \cong \mathrm{Hom}_R(A,X)$

(Proof)

$P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} A \to 0$ exact $\Rightarrow$ $0 \to \mathrm{Hom}_R(A,X) \xrightarrow{d_0} \mathrm{Hom}_R(P_0,X) \xrightarrow{d_1} \mathrm{Hom}(P_1,X)$ is exact

$\Rightarrow \mathrm{Ext}^0_R(A,X) = \ker(d_1) \cong \mathrm{im}(d_0) \cong \mathrm{Hom}_R(A,X)$ $\quad$ (using injectivity of d_0)

Ex: $R = \mathbb{Z}$ $\quad A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$ $\quad X =$ any $\mathbb{Z}$-mod. $\left(= \text{abel. group}\right)$

Proj res: $\cdots \to 0 \to \mathbb{Z} \xrightarrow{6} \mathbb{Z} \to \mathbb{Z}/6\mathbb{Z} \to 0$

Hom

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}, X) \xrightarrow{d_0} \underset{\substack{\cong \\ X}}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},X)} \xrightarrow{d_1} \underset{\substack{\cong \\ X}}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},X)} \xrightarrow{d_2} 0 \to 0 \to \cdots$$

$$X \xrightarrow{\times 6} X$$

$\mathrm{Ext}^0_{\mathbb{Z}}(A,X) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}, X)$

$\mathrm{Ext}^1_{\mathbb{Z}}(A,X) \cong \ker(d_2)/\mathrm{im}(d_1) \cong X/6X$

$\mathrm{Ext}^n_{\mathbb{Z}}(A,X) = 0$ for $n \geq 2$

Check that we get the same groups using diff. proj. res.

Prop: Given a short exact seq. of R-modules $0 \to A \to B \to C \to 0$ and projective resolutions (P_i) of A and $(\overline{P}_i)$ of C, there is a proj. res. of B given by $(P_i \oplus \overline{P}_i)$ s.t. the following diagram commutes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_2 \arrow[r] \arrow[d] & P_2 \oplus \overline{P}_2 \arrow[r] \arrow[d] & \overline{P}_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_1 \oplus \overline{P}_1 \arrow[r] \arrow[d] & \overline{P}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_0 \oplus \overline{P}_0 \arrow[r] \arrow[d] & \overline{P}_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Taking $\mathrm{Hom}_R(-,X)$ of this gives a short exact sequence of cochain complexes:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \mathrm{Hom}(\overline{P}_1,X) \arrow[r] \arrow[u] & \mathrm{Hom}(P_1 \oplus \overline{P}_1,X) \arrow[r] \arrow[u] & \mathrm{Hom}(P_1,X) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\overline{P}_0,X) \arrow[r] \arrow[u] & \mathrm{Hom}(P_0 \oplus \overline{P}_0,X) \arrow[r] \arrow[u] & \mathrm{Hom}(P_0,X) \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Gives a long exact sequence of cohomology groups

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(C,X) \arrow[r] & \mathrm{Hom}_R(B,X) \arrow[r] & \mathrm{Hom}(A,X) \arrow[dll] \\
 & \underline{\mathrm{Ext}^1_R(C,X)} \arrow[r] & \mathrm{Ext}^1_R(B,X) \arrow[r] & \mathrm{Ext}^1_R(A,X) \arrow[dll] \\
 & \mathrm{Ext}^2_R(C,X) \arrow[r] & \cdots &
\end{tikzcd}$$

measures failure of $\mathrm{Hom}(-,X)$ to be exact

$\boxed{\mathrm{Hom}_R(X,-)}$

$$\begin{tikzcd} A \arrow[r] & B \\ X \arrow[u] \arrow[ur] & \end{tikzcd}$$

Thm: Given an exact sequence $0 \to A \to B \to C \to 0$, there is a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R(X,A) \arrow[r] & \mathrm{Hom}_R(X,B) \arrow[r] & \mathrm{Hom}_R(X,C) \arrow[dll] \\ & \mathrm{Ext}^1_R(X,A) \arrow[r] & \mathrm{Ext}^1_R(X,B) \arrow[r] & \mathrm{Ext}^1_R(X,C) \arrow[dll] \\ & \mathrm{Ext}^2_R(X,A) \arrow[r] & \cdots & \end{tikzcd}$$

$\boxed{X \otimes_R -}$ $\rightsquigarrow$ $\mathrm{Tor}(X,A)$ (next time)