S_3

		id	(12)	(123)
φ_1	χ_1	1	1	1
φ_2	χ_2	1	-1	1
φ_3	χ_3	2	0	-1

$\varphi_3 : S_3 \to GL_2(\mathbb{C})$

$\varphi_3 \otimes \varphi_3 : S_3 \to GL_4(\mathbb{C})$

$\varphi_3 \otimes \varphi_3$ char: $\chi_3 \chi_3 = \chi$

$$\chi(id) = 4 \qquad \chi((12)) = 0 \qquad \chi((123)) = 1$$

What is the decomposition of χ into irreducible characters?

$$\langle \chi, \chi_i \rangle = \frac{1}{|S_3|}\left[\chi(id)\overline{\chi_i(id)} + 3\chi((12))\overline{\chi_i((12))} + 2\chi((123))\overline{\chi_i((123))}\right]$$

$$i=1 \to \frac{1}{6}\left[4\cdot 1 + 3\cdot 0\cdot 1 + 2\cdot 1\cdot 1\right] = 1$$

$$i=2 \to \frac{1}{6}\left[4\cdot 1 + 3\cdot 0\cdot(-1) + 2\cdot 1\cdot 1\right] = 1$$

$$i=3 \to \frac{1}{6}\left[4\cdot 2 + 3\cdot 0\cdot 0 + 2\cdot 1\cdot(-1)\right] = 1$$

$$\chi = \chi_1 + \chi_2 + \chi_3$$

Homological Alg. definitions:

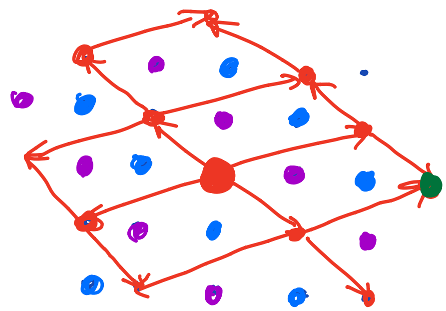
$H_n(A,B)$, $H^n(A,B)$, Ext, Tor, long exact sequence thms.

You don't have to memorize Prop. 13 from §18.3 but you should know what a primitive central idempotent is.

Midterm #2 3e)

$R = \mathbb{Z}, \quad M = \mathbb{Z}^2/N \qquad N \subseteq \mathbb{Z}^2$ submod. gen. by $\{(2,1), (1,-1)\}$

Find rank, invariant factors, elementary factors



$N = \{a(2,1) + b(1,-1) : a, b \in \mathbb{Z}\}$

$(3,0) = (2,1) + (1,-1)$

$(3,0), \ (1,-1)$

$\underline{(1,0)} + N \in M$

$3(1,0) + N = N$

$(2,0) + N$

$M = \mathbb{Z}^2/N \cong \mathbb{Z}/3\mathbb{Z}$

rank $= 0$ inv. fact $=$ elem. factor $= \{3\}$

Hunting for irred. representations

G finite group. Immediate reps: trivial (1-dim'l)

regular (-dim'l)

What?! [rep. corresponding to $\mathbb{C}G$ as mod. over itself.

M is a $\mathbb{C}G$-module

What is corresponding rep. of G?

M is a $\mathbb{C}$-module $=$ vec. over $\mathbb{C}$

Elements of $\mathbb{C}G$ act on M linearly.

$G \to GL(M)$ (group hom)

$\mathbb{C}G \to \mathrm{Hom}_{\mathbb{C}}(M,M)$ (ring hom)

dim of rep $= \dim_{\mathbb{C}} M$

Regular representation : $M = \underline{\underline{\mathbb{C}G}}$ $\leftarrow \dim_{\mathbb{C}} = |G|$

Basis for $\mathbb{C}G = \{ g \in G \}$ (as $\mathbb{C}$-vec. space)

Ex: $S_3 = G$ $\quad \mathbb{C}S_3 = \{ \underbrace{a_1\, id + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)}_{a_1,\dots,a_6 \in \mathbb{C}} \}$

$G \to GL_6(\mathbb{C})$

$(12) \longmapsto ?$ $\quad (12)(\quad) = a_1(12) + a_2\, id + a_3(132) + a_4(123) + a_5(23) + a_6(13)$

$(13) = (12)(132)$

$(12)(13) = (132)$

$(12)(23) = (123)$

$(23) = (12)(123)$

$$(12) \to \begin{array}{c|cccccc} & id & (12) & (13) & (23) & (123) & (132) \\ \hline id & 0 & 1 & 0 & 0 & 0 & 0 \\ (12) & 1 & 0 & 0 & 0 & 0 & 0 \\ (13) & 0 & 0 & 0 & 0 & 0 & 1 \\ (23) & 0 & 0 & 0 & 0 & 1 & 0 \\ (123) & 0 & 0 & 0 & 1 & 0 & 0 \\ (132) & 0 & 0 & 1 & 0 & 0 & 0 \end{array}$$

φ = reg. rep. $\quad \rho$ = character of φ

$\rho(id) = |G|$ $\quad \underset{g \neq id}{\rho(g)} = \#\{ h \in G : \underbrace{gh = h}_{\text{mult on right by } h^{-1} \;\to\; g = id} \} = 0$

How does φ decompose into irreducibles?

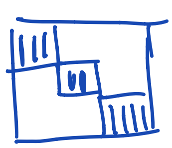
$$\langle \rho, \chi_i \rangle = \frac{1}{|G|} \cdot \rho(id)\overline{\chi_i(id)} = \frac{1}{|G|} \cdot |G| \cdot \overline{\chi_i(id)}$$

$$= \overline{\chi_i(id)}$$

= dim of i^{th} irred. rep.

$\mathbb{C}G \cong \underbrace{M_{n_1}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C})}$

n_1 copies of 1^{st} irred rep
(n_1 columns)



$R = M_{n_1}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C})$

Primitive idempotents E_{ii}

$L_i = RE_{ii}$

$$\begin{bmatrix} ||| & & \\ & || & \\ & & |||| \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}}_{E_{11}} = \begin{bmatrix} \begin{matrix} | & 0 \end{matrix} & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$= \{A \in R : \text{cols } 2 \to n = 0\}$

Claim: L_i irreducible

Claim: L_i, L_j isomorphic if they come from same block.

$\begin{bmatrix} v & 0 \end{bmatrix}^{\neq 0} \in RE_{11}$ there exist $A \in R$ s.t. $A[v\ 0] = E_{11}$ $\Rightarrow$ only submodule of RE_{11} is $\{0\}$

$\Rightarrow R[v\ 0] = RE_{11}$

What are primitive central idempotents? $(0, \dots, 0, I_{n_i}, 0, \dots, 0) = e_i$

$Re_i = 0 \times \dots \times M_{n_i}(\mathbb{C}) \times 0 \times \dots \times 0$

Ex: S_3 Prim. central idemp. $z_3 = \dfrac{\chi_3(id)}{|G|} \sum_{g \in G} \chi_3(g^{-1})g$

$$z_3 = \frac{2}{6}\left[2 \cdot id - (123) - (132)\right]$$

$(\mathbb{C}S_3)z_3 \ni$

$2\,id - (123) - (132),\ = v_1$

$2(12) - (23) - (13),\ = v_2$

$2(13) - (12) - (23),\ = v_3$

$2(23) - (13) - (12) = -v_2 - v_3$

$2(123) - (132) - id = v_4$

$2(132) - id - (123) = -v_1 - v_4$

4 dim'l over $\mathbb{C}$ (just like $M_2(\mathbb{C})$)

$= \left\{ a_1 id + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132) : \begin{matrix} a_1 + a_5 + a_6 = 0 \\ a_2 + a_3 + a_4 = 0 \end{matrix} \right\}$

subring of $\mathbb{C}S_3$ gen. by prim. central idempotent. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

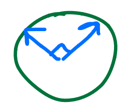
How can I find the primitive idempotents? $\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix}$

Q: Is a choice of primitive idempotents e_1, e_2 s.t. $e_1e_2 = e_2e_1 = 0$ and $e_1 + e_2 = id$ unique? NO

e.g. $\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix} = P \qquad P \cdot P = \left(\frac{1}{2}\right)^2\begin{pmatrix}2&2\\2&2\end{pmatrix} = P$

$$(I_2 - P)(I_2 - P) = I_2 - 2P + P^2$$
$$= I_2 - 2P + P = I_2 - P$$
$$P(I_2 - P) = P - P^2 = 0$$

Depends only on $P^2 = P$



$\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$

orth. primitive idem.