Welcome to MA 721 Office Hours

Q: Will there be algebraic geometry on exam?

A: No. Motivational.

" algebraic sets
Hilbert's Nullstellensatz
Krull dim

Q: Will there be primary decompositions?

A: Yes!

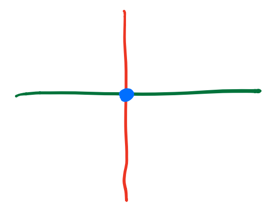
Q: Artinian?

A: No!

Ex: $\langle x \rangle \cap \langle y \rangle \cap \langle x^3, x^2y, xy^2, y^3 \rangle$

$\langle x \rangle$ $\quad \langle y \rangle$ $\quad \langle x, y \rangle$

embedded



Should be able to
convert Jordan can. form $\longleftrightarrow$ rational can. form

Ex: $\mathbb{Q}[x]/\langle x^2(x-1)\rangle \oplus \mathbb{Q}[x]/\langle x(x-1)^2\rangle$ $\leftarrow$ a $\mathbb{Q}[x]$-module

Q: What are invariant factors?
elementary divisors? $\leftarrow$

Q': What is rational can. form of linear transf. given by mult. by x?
" " Jordan can. form " " ?

$$\mathbb{Q}[x]/\langle x^2(x-1)\rangle \cong \mathbb{Q}[x]/\langle x^2\rangle \oplus \mathbb{Q}[x]/\langle x-1\rangle$$

$$\mathbb{Q}[x]/\langle x(x-1)^2\rangle \cong \mathbb{Q}[x]/\langle x\rangle \oplus \mathbb{Q}[x]/\langle (x-1)^2\rangle$$

$$\mathbb{Q}[x]/\langle x^2(x-1)\rangle \oplus \mathbb{Q}[x]/\langle x(x-1)^2\rangle \cong \mathbb{Q}[x]/\langle x\rangle \oplus \mathbb{Q}[x]/\langle x^2\rangle \oplus \mathbb{Q}[x]/\langle x-1\rangle \oplus \mathbb{Q}[x]/\langle (x-1)^2\rangle$$

Elem. divisors: $x, x^2, x-1, (x-1)^2$

$R/\langle p_1^{\alpha_1}\rangle \oplus \dots \oplus R/\langle p_t^{\alpha_t}\rangle$ $\quad p_i$ prime

Invariant factors: $a_1 = x(x-1)$ $\quad a_2 = x^2(x-1)^2$

$R/\langle a_1\rangle \oplus \dots \oplus R/\langle a_m\rangle$

where $a_1 \mid a_2 \mid \dots \mid a_m$

$$\underbrace{\mathbb{Q}[x]/\langle x(x-1)\rangle \oplus \mathbb{Q}[x]/\langle x^2(x-1)^2\rangle}_{6\text{-dim'l } \mathbb{Q}\text{-vec. space}}$$

Jordan can. form of linear transf. given by mult. by x

$$\begin{pmatrix} 0 & & & & & \\ & 0 & 1 & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}$$

Elem. div.		Jordan block
x	$\longrightarrow$	eig val 0, size 1
x^2	$\longrightarrow$	eig val 0, size 2
$x-1$	$\longrightarrow$	eigval 1, size 1
$(x-1)^2$		eigval 1, size 2

$\mathbb{Q}[x]/\langle (x-\lambda)^k\rangle$ $\quad \{(x-\lambda)^{k-1}, (x-\lambda)^{k-2}, \dots, (x-\lambda), 1\}$

use this basis for Jordan can. form.

Elem divisors: $p_1^{\alpha_{11}}, \dots, \underline{p_1^{\alpha_{1d_1}}}$,

$p_2^{\alpha_{21}}, \dots, \underline{p_2^{\alpha_{2d_2}}}$

$\vdots \qquad \vdots$

$p_k^{\alpha_{k1}}, \dots, \underline{p_k^{\alpha_{kd_k}}}$

$p_i \neq p_j$ distinct primes

$\alpha_{ij} < \alpha_{i(j+1)}$

What are the invariant factors?

$$R/\langle p_1^{\alpha_{11}}\rangle \oplus \dots \oplus \underline{R/\langle p^{\alpha_{1d_1}}\rangle} \oplus \dots \oplus R/\langle p_k^{\alpha_{k1}}\rangle \oplus \dots \oplus \underline{R/\langle p_k^{\alpha_{kd_k}}\rangle}$$

$a_1 \mid a_2 \mid \dots \mid a_m$

$a_m = p_1^{\alpha_{1d_1}} \cdots p_k^{\alpha_{kd_k}}$

$\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \not\cong \mathbb{Z}/25\mathbb{Z}$ $\qquad m = ?$

$\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \cong \mathbb{Z}/35\mathbb{Z}$ $\qquad \max\{d_1, \dots, d_k\}$

$a_{m-1} = p_1^{\alpha_{1(d_1-1)}} \cdots p_k^{\alpha_{k(d_k-1)}}$

Ex: Elem divisors over $\mathbb{Z}$:

$$\begin{array}{l} 2,\ 2,\ 2^2,\ 2^5 \\ 3,\ 3^7 \\ 5,\ 5,\ 5^2 \\ 7,\ 7,\ 7^3 \end{array}$$

right justify, pad with ones

rel prime

Invariant factors:

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \dots \quad \dots \oplus \mathbb{Z}/7^3\mathbb{Z}$$

$$\cong \mathbb{Z}/a_1\mathbb{Z} \oplus \mathbb{Z}/a_2\mathbb{Z} \oplus \mathbb{Z}/a_3\mathbb{Z} \oplus \mathbb{Z}/a_4\mathbb{Z}$$

$$\begin{array}{|c|c|c|c|} \hline 2 & 2 & 2^2 & 2^5 \\ 1 & 1 & 3 & 3^7 \\ 1 & 5 & 5 & 5^2 \\ 1 & 7 & 7 & 7^3 \\ \hline \end{array}$$

$a_1 = 2$ $\quad a_2 = 2 \cdot 5 \cdot 7$ $\quad a_3 = 2^2 \cdot 3 \cdot 5 \cdot 7$ $\quad a_4 = 2^5 \cdot 3^7 \cdot 5^2 \cdot 7^3$

Know def of Torsion submodule & Annihilator

R PID, M finitely. gen. R-mod

$$\Rightarrow \quad R = R^r \oplus \underbrace{R/\langle a_1\rangle \oplus \dots \oplus R/\langle a_m\rangle}$$

where $a_1 | a_2 | \dots | a_m$

$$\mathrm{Tor}(M) = \{0\}^r \oplus R/\langle a_1\rangle \oplus \dots \oplus R/\langle a_m\rangle$$

$$\mathrm{Ann}(\mathrm{Tor}(M)) = \langle a_m\rangle = \langle r \in R : r \cdot m = 0 \text{ for } m \in \mathrm{Tor}(M)\rangle$$

$$\mathrm{rank}(M) = r$$

From Worksheet

$$A = \begin{pmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

rational canonical form of $\mathbb{C}[x]/\langle x^4-4x^2+4\rangle$

represents mult. by x in $\mathbb{C}[x]/\langle x^4-4x^2+4\rangle$ w.r.t. $\{1, x, x^2, x^3\}$

$$x^4-4x^2+4 = (x^2-2)^2 = (x-\sqrt{2})^2(x+\sqrt{2})^2$$

$$\mathbb{C}[x]/\langle x^4-4x^2+4\rangle \cong \mathbb{C}[x]/\langle (x-\sqrt{2})^2\rangle \oplus \mathbb{C}[x]/\langle (x+\sqrt{2})^2\rangle$$

$$J = \left(\begin{array}{cc|cc} \sqrt{2} & 1 & & \\ 0 & \sqrt{2} & & \\ \hline & & -\sqrt{2} & 1 \\ & & 0 & -\sqrt{2} \end{array}\right)$$

represents mult. by x in $\mathbb{C}[x]/\langle (x-\sqrt{2})^2\rangle \oplus \mathbb{C}[x]/\langle (x+\sqrt{2})^2\rangle$ w.r.t. the basis $\{(x-\sqrt{2},0), (1,0), (0,x+\sqrt{2}), (0,1)\}$

$$f \in \mathbb{C}[x]/\langle x^4-4x^2+4\rangle \longrightarrow \underline{(f \bmod I_1, f \bmod I_2)} \in \mathbb{C}[x]/\underbrace{\langle (x-\sqrt{2})^2\rangle}_{I_1} \oplus \mathbb{C}[x]/\underbrace{\langle (x+\sqrt{2})^2\rangle}_{I_2}$$

$$1 \in \mathbb{C}[x]/\langle x^4-4x^2+4\rangle \longrightarrow (1, 1)$$

$$= \underline{0}\cdot(x-\sqrt{2},0) + \underline{1}\cdot(1,0) + \underline{0}(0, x+\sqrt{2}) + \underline{1}\cdot(0,1)$$

$$x \in \mathbb{C}[x]/\langle x^4-4x^2+4\rangle \longrightarrow (x, x)$$

$$= \underline{1}\cdot(x-\sqrt{2},0) + \underline{\sqrt{2}}(1,0) + \underline{1}(0, x+\sqrt{2}) \underline{-\sqrt{2}}\cdot(0,1)$$

$$x^2 \in \quad " \quad " \longrightarrow (x^2, x^2)$$

$x^2 = \underline{2\sqrt{2}x-2} \mod I_1$

$$\| $$

$$(2\sqrt{2}x-2, -2\sqrt{2}x-2) \qquad 0 = (x-\sqrt{2})^2 = x^2 - 2\sqrt{2}x + 2$$

$$= \underline{\quad}(x-\sqrt{2},0) + \underline{\quad}(1,0) + \underline{\quad}(0,x+\sqrt{2}) + \underline{\quad}(0,1)$$

$x^3 = $ " . . . $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$

15.2.32 Q_1, Q_2 primary $\downarrow$ M-primary $\quad \sqrt{Q_1} = M \quad \sqrt{Q_2} = M \quad$ M maximal

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$\Rightarrow Q_1 + Q_2$ primary $\quad \sqrt{Q_1+Q_2} = M$

$Q_1 \cdot Q_2$ primary $\quad \sqrt{Q_1 \cdot Q_2} = M$

p 682 "belonging"

$Q_1 \subseteq Q_1 + Q_2$

$M = \sqrt{Q_1} \subseteq \sqrt{Q_1+Q_2}$

$I : \langle b \rangle = \{r \in R : rb \in I\}$

• $(Q_1+Q_2):\langle b \rangle$ primary

$Q_1 + Q_2$ primary?

$a \cdot b \in Q_1 + Q_2 \quad b \notin Q_1 + Q_2$

$a \cdot b = q_1 + q_2$ WTS $a \in \sqrt{Q_1+Q_2}$

i.e. $a^k \in Q_1 + Q_2$ for some k

• $a \cdot b - q_2 \in Q_1$?

R/Q_1

Seems tricky.

Ideas: $Q_1 + Q_2 = \bigcap_{j=1}^{m} I_j \quad I_j$ primary ??

$M^{n_1} \subseteq Q_1 \subseteq M$

$M^{n_2} \subseteq Q_2 \subseteq M$

$M^{n_i} \subseteq Q_i \subseteq Q_1 + Q_2 \subseteq M$?

p 682

Q is P-primary $\Rightarrow P^m \subseteq Q \subseteq P$

M maximal, $M^n \subseteq Q \subseteq M \Rightarrow Q$ is M-primary

$$\begin{array}{c} M^{n_1} \cdot M^{n_2} = M^{n_1+n_2} \\ \cap\!\mid \\ Q_1 \cdot Q_2 \\ \cap\!\mid \; ? \\ M \end{array}$$