Math 721 – Worksheet 01/14/20

(10:15am) Form into 4 groups of size 3-4.
(10:17am) As a group, read through the definition of tensor product and try Problem 0.
(10:30am) Discuss as a class.
(10:35am) Students in Group X work on Problem X.
          If it’s too hard, try the warm-up. If you finish, try another problem.
(11:05am) Groups present their findings. (∼5 minutes per group)

Let $R$ be a commutative ring with $1 \neq 0$ and suppose $M$ and $N$ are both (left) $R$-modules.

**Definition.** The tensor product of $M$ and $N$ over $R$, $M \otimes_R N$, is the set of formal sums

$$M \otimes_R N = \left\{ \sum_{i=1}^{k} m_i \otimes n_i : k \in \mathbb{Z}_{\geq 0}, m_i \in M, n_i \in N \right\}$$

subject to the relations

$$(m_1 + m_2) \otimes n = (m_1 \otimes n) + (m_2 \otimes n),$$

$$m \otimes (n_1 + n_2) = (m \otimes n_1) + (m \otimes n_2),$$

$$rm \otimes n = m \otimes rn$$

for all $m_1, m_2, m \in M$, $n_1, n_2, n \in N$ and $r \in R$. This is an $R$-module under the action

$$r \left( \sum_{i=1}^{k} m_i \otimes n_i \right) = \sum_{i=1}^{k} rm_i \otimes n_i = \sum_{i=1}^{k} m_i \otimes rn_i.$$

**Example.** Consider $R = \mathbb{R}$ and $M = N = \mathbb{R}^2 = \{ ae_1 + be_2 : a, b \in \mathbb{R} \}$. The tensor product $\mathbb{R}^2 \otimes_\mathbb{R} \mathbb{R}^2$ is a 4-dimensional vectorspace with basis

$$\{ e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2 \}.$$

**Problem 0.** Write the element $(e_1 + e_2) \otimes (e_1 + 2e_2) + (e_1 - e_2) \otimes (3e_1)$ of $\mathbb{R}^2 \otimes_\mathbb{R} \mathbb{R}^2$ as an $\mathbb{R}$-linear combination of the basis above.

Show each of the following $R$-module isomorphisms:

**Warm-up**

**Problem 1.** $M \otimes_R R \cong M$ $R \otimes_R R \cong R$

**Problem 2.** $R^m \otimes_R R^n \cong R^{mn}$ $m = 2$ and $n = 2$

**Problem 3.** $Q^2 \otimes_Q R \cong R^2$ $Q \otimes_Q R \cong R$

**Problem 4.** $\mathbb{Z}/m\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ where $d = \gcd(m, n)$ $m = 2$ and $n = 2, 3$