Math 721 – Worksheet 04/21/20

Lets find irreducible characters of the symmetric group S_4.

Problem 0. Find all the conjugacy classes of S_4 and their sizes.

Problem 1 (Numerology)**.**

(a) What are all of the 1-dimensional complex representations of S_4?

(b) What are the dimensions of each of the irreducible representations of S_4?

Problem 2. Consider the permutation representation $\pi \mapsto P_\pi \in \mathrm{GL}_4(\mathbb{C})$ of S_4.

(a) What is the character of this representation?

(b) Note that the subspace $\mathbb{C}\{(1,1,1,1)\}$ is fixed by every group element. What is the character of the representation restricted to this subspace?

(c) What is the character χ of the representation of S_4 induced by restricting the permutation representation to the subspace $\{x \in \mathbb{C}^4 : \sum_{i=1}^4 x_i = 0\}$?

(d) Find $\langle \chi, \chi \rangle$ to verify that χ is irreducible.

Problem 3. Let V denote the normal subgroup $\{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$ of S_4.

(a) Show that the quotient group S_4/V is isomorphic to S_3.

Hint: consider the action of S_4 on $V\backslash\{\mathrm{id}\} = \{(12)(34), (13)(24), (14)(23)\}$ given by conjugation.

(b) Find the character χ of S_4 induced by the map $S_4 \to S_4/V \cong S_3 \xrightarrow{\varphi} \mathrm{GL}_2(\mathbb{C})$ where φ is the irreducible 2-dimensional representation of S_3.

(c) Find $\langle \chi, \chi \rangle$ to verify that χ is irreducible.

Solutions

Problem 0 Solution. Recall that the conjugacy classes in S_4 are determined by their cycle structure. From this we see that there are five:

representative element:	id	(12)	(123)	(1234)	(12)(34)
size:	1	6	8	6	3

Problem 1 Solution. a) There are two 1-dimensional complex representations of S_4, the trivial representation and the sign representation. To see this, let $\varphi : S_4 \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$ be a group homomorphism. Then $\varphi(\mathrm{id}) = 1$ and $1 = \varphi(\mathrm{id}) = \varphi((12)(12)) = \varphi(12)^2$, so $\varphi((12)) \in \{\pm 1\}$. Moreover, since φ is a group homomorphism and every element of S_4 is generated by transpositions, the value $\varphi((12))$ determines the value of φ on all elements of S_4.

b) Since there are five conjugacy classes in S_4, there are five irreducible representations. If they have dimensions $n_1 \le n_2 \le n_3 \le n_4 \le n_5$, then by part (a), $n_1 = n_2 = 1$ and $2 \le n_3$. We know that $\sum_{j=1}^5 n_j^2 = |S_4| = 24$ and so $n_3^2 + n_4^2 + n_5^2 = 22$. From this we see that $n_3 = 2$ and $n_4 = n_5 = 3$.

Problem 2 Solution. a) Note that the character of this representation sends $\pi \in S_4$ to $\mathrm{trace}(P_\pi)$, which is the number of fixed points of π. Explicitly,

conj. class:	id	(12)	(123)	(1234)	(12)(34)
character:	4	2	1	0	0

b) Let $v = (1,1,1,1)^T$. Since $P_\pi v = v$ for all $\pi \in S_4$, we see that P_π acts as the identity on this 1-dimensional subspace. The corresponding character is the trivial character.

c) Note that $\mathbb{C}^4$ as a $\mathbb{C}S_4$ module under the permutation representation is a direct sum of the submodules $\mathbb{C}v$ and $v^\perp = \{x \in \mathbb{C}^4 : \sum_{i=1}^4 x_i = 0\}$. Therefore the character χ_{perm} of the permutation representation is the sum of the characters χ_{trivial} and χ for the restrictions to these submodules. Therefore $\chi = \chi_{\mathrm{perm}} - \chi_{\mathrm{trivial}}$, giving

$$\chi(\pi) = \chi_{\mathrm{perm}}(\pi) - \chi_{\mathrm{trivial}}(\pi) = \#\{\text{fixed points of } \pi\} - 1.$$

Explicitly, it takes values

conj. class:	id	(12)	(123)	(1234)	(12)(34)
χ:	3	1	0	-1	-1

d) Note that $\langle \chi, \chi \rangle = \frac{1}{24} \sum_{i=1}^5 |K_i| \chi(\pi_i) \overline{\chi(\pi_i)}$ where $\pi_i \in K_i$. From this we calculate

$$\begin{aligned}
\langle \chi, \chi \rangle &= \frac{1}{24}\left(1 \cdot \chi(\mathrm{id})^2 + 6 \cdot \chi((12))^2 + 8 \cdot \chi((123))^2 + 6 \cdot \chi((1234))^2 + 3 \cdot \chi((12)(34))^2\right) \\
&= \frac{1}{24}\left(1 \cdot 3^2 + 6 \cdot 1^2 + 8 \cdot 0 + 6 \cdot (-1)^2 + 3 \cdot (-1)^2\right) \\
&= \frac{1}{24}(9 + 6 + 6 + 3) = 1
\end{aligned}$$

Since $\langle \chi, \chi \rangle = 1$, the character χ is irreducible.

Problem 3 Solution. a) Since $V\backslash\{\mathrm{id}\}$ is conjugacy class, S_4 naturally acts on it by conjugation. It is not hard to check that the kernel of this action (i.e. the set of group elements fixing each of $V\backslash\{\mathrm{id}\}$) is the normal subgroup V. This gives an injective group homomorphism $S_4/V \to S_{V\backslash\{\mathrm{id}\}} \cong S_3$. Since these groups both have size $6 = 24/4$, it must be an isomorphism.

b) Recall that the 2-dimensional representation ψ of S_3 is given by $\psi(\mathrm{id}) = 2$, $\psi((12)) = 0$, and $\psi((123)) = -1$. For a representative of each conjugacy class in S_4, we need to calculate its image in S_3.

Note that id and $(12)(34)$ belong to V, and so are sent to the identity element in $S_4/V \cong S_3$, giving $\chi(\mathrm{id}) = \chi((12)(34)) = \psi(\mathrm{id}) = 2$.

We see that (12) acts on the set $\{(12)(34), (13)(24), (14)(23)\}$ by fixing the first element and switching the second and third. Therefore $\chi((12)) = \psi((23)) = 0$.

Similarly, the 3-cycle (123) acts on this set by

$$(12)(34) \mapsto (23)(14) = (14)(23), \quad (13)(24) \mapsto (21)(34) = (12)(34), \quad (14)(23) \mapsto (24)(31) = (13)(24),$$

giving $\chi((123)) = \psi((132)) = -1$.

Finally, we note that the 4-cycle (1234) acts on $V\backslash\{\mathrm{id}\}$ by

$$(12)(34) \mapsto (23)(41) = (14)(23), \quad (13)(24) \mapsto (24)(13) = (13)(24), \quad (14)(23) \mapsto (21)(34) = (12)(34),$$

giving $\chi((1234)) = \psi((13)) = 0$.

All together this gives:

conj. class:	id	(12)	(123)	(1234)	(12)(34)
χ:	2	0	-1	0	2

c) Note that $\langle \chi, \chi \rangle = \frac{1}{24} \sum_{i=1}^{5} |K_i| \chi(\pi_i) \overline{\chi(\pi_i)}$ where $\pi_i \in K_i$. From this we calculate

$$\begin{aligned}
\langle \chi, \chi \rangle &= \frac{1}{24} \left(1 \cdot \chi(\mathrm{id})^2 + 6 \cdot \chi((12))^2 + 8 \cdot \chi((123))^2 + 6 \cdot \chi((1234))^2 + 3 \cdot \chi((12)(34))^2 \right) \\
&= \frac{1}{24} \left(1 \cdot 2^2 + 6 \cdot 0^2 + 8 \cdot (-1)^2 + 6 \cdot 0^2 + 3 \cdot 2^2 \right) \\
&= \frac{1}{24} (4 + 8 + 12) = 1
\end{aligned}$$

Since $\langle \chi, \chi \rangle = 1$, the character χ is irreducible.