Math 721 – Homework 9

Due Friday, April 10 at 5pm

Good practice problems (do not turn in solutions):
DF 17.1: 12, 14, 19, 27
DF 17.2: 4, 5, 6, 8, 20

Problem 1 (DF 17.1.21)**.** Let $R = k[x,y]$ where k is a field and let I denote the maximal ideal $\langle x, y\rangle$ in R.

(a) Show that the following is a free resolution of k as an R-module:

$$0 \to R \xrightarrow{\alpha} R^2 \xrightarrow{\beta} R \xrightarrow{\pi} k \to 0,$$

where

$$\alpha(f) = (yf, -xf), \quad \beta(f,g) = xf + yg, \quad \text{and} \quad \pi(f) = f + I \in R/I \cong k.$$

(b) Use the resolution in (a) to show that $\mathrm{Tor}_2^R(k,k) \cong k$.

(c) Prove that $\mathrm{Tor}_1^R(k,I) \cong k$. (Hint: Use the long exact sequence corresponding to the short exact sequence $0 \to I \to R \to k \to 0$ and (b).)

(d) Conclude that the torsion-free R module I is not flat.

Problem 2 (DF 17.2.9 +)**.** Let G be an infinite cyclic group with generator σ.

(a) Show that the map $\mathrm{aug} : \mathbb{Z}G \to \mathbb{Z}$ defined by

$$\mathrm{aug}\left(\sum_{i\in\mathbb{Z}} a_i \sigma^i\right) = \sum_{i\in\mathbb{Z}} a_i$$

is a $(\mathbb{Z}G)$-module homomorphism, taking the trivial action of G on $\mathbb{Z}$.

(b) Prove that multiplication by $\sigma - 1$ in $\mathbb{Z}G$ gives the following free resolution of $\mathbb{Z}$ as a $(\mathbb{Z}G)$-module:

$$0 \to \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{\mathrm{aug}} \mathbb{Z} \to 0.$$

(c) Let A be a G-module. Show that $H^0(G,A) \cong A^G$, $H^1(G,A) \cong A/(\sigma-1)A$, and $H^n(G,A) = 0$ for $n \geq 2$.

(d) Show that $H^1(G, \mathbb{Z}G) \cong \mathbb{Z}$.
(This shows that free modules can have nontrivial cohomology groups.)