Math 721 – Homework 8

Due Friday, April 3 at 5pm on Moodle

Good practice problems (do not turn in solutions):
DF 17.1: 2, 4, 5, 7, 8

Problem 1 (DF 17.1 Exercise 1)**.** Give the details of the proof of the following proposition:

Proposition 17.1.1. A homomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ of cochain complexes induces group homomorphisms from $H^n(\mathcal{A}) \to H^n(\mathcal{B})$ for $n \geq 0$ on their respective cohomology groups.

Problem 2 (DF 17.1 Exercise 3)**.** Suppose

$$\begin{array}{ccccccc} & & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B & \xrightarrow{\beta'} & C & & \end{array}$$

is a commutative diagram of R-modules with exact rows. Show the following:

(a) If $c \in \ker(h)$ and $\beta(b) = c$, then $g(b) \in \ker(\beta')$ and $g(b) = \alpha'(a')$ for some $a' \in A'$.

(b) The map $\delta : \ker(h) \to A'/\operatorname{image}(f)$ given by $\delta(c) = a' \mod \operatorname{image}(f)$ is a well-defined R-module homomorphism.

(c) (*The Snake Lemma*) There is an exact sequence

$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\delta} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h),$$

where $\operatorname{coker}(f) = A'/\operatorname{image}(f)$ denotes the *cokernel* of f and similarly for g and h.

(d) If α is injective and β' is surjective (i.e. the two rows in the above commutative diagram can be extended to short exact sequences) then the sequence in (c) can be extended to an exact sequence

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\delta} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h) \to 0.$$