Math 721 – Homework 7 Solutions

Problem 1 (DF 15.1 Exercise 2)**.** Show that each of the following rings are *not* Noetherian by exhibiting an explicit infinite increasing chain of ideals.

(a) the ring of continuous real valued functions on $[0, 1]$,

(b) the ring of all functions from $\mathbb{N}$ to $\mathbb{Z}/2\mathbb{Z}$.

Proof of (a). Let R denote the ring of continuous real valued functions on $[0, 1]$. For each $n \in \mathbb{Z}_+$, let

$$I_n = \{f \in R : f(x) = 0 \text{ for all } x \in [0, 1/n]\}.$$

We can check that I_n is an ideal of R. It is nonzero, since it contains the zero function. For any $f_1, f_2 \in I_n$ and $g \in R$, we can check that for $x \in [0, 1/n]$,

$$(f_1 + gf_2)(x) = f_1(x) + g(x)f_2(x) = 0 + g(x) \cdot 0 = 0.$$

Therefore $f_1 + gf_2 \in I_n$ and I_n is an ideal. Moreover, since $[0, 1/(n+1)]$ is a subset of $[0, 1/n]$, I_n is a subset of I_{n+1}. Moreover, for each n, we can define a continuous function

$$f_n(x) = \begin{cases} 0 & \text{if } x \in [0, 1/n] \\ x - 1/n & \text{if } x \in (1/n, 1]. \end{cases}$$

Then f_{n+1} belongs to I_{n+1}, but not I_n. Therefore $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$ is an infinite ascending chain of ideals that does not terminate. This shows that the ring R is not Noetherian. □

Proof of (b). Let R denote the ring of the ring of all functions from $\mathbb{N}$ to $\mathbb{Z}/2\mathbb{Z}$ For each $n \in \mathbb{Z}_+$, let

$$I_n = \{f \in R : f(x) = 0 \text{ for all } x \geq n\}.$$

As in part (a), we can check that I_n is an ideal of R. It contains the zero function and for any $f_1, f_2 \in I_n$ and $g \in R$, and for any $x \in \mathbb{N}$ with $x \geq n$,

$$(f_1 + gf_2)(x) = f_1(x) + g(x)f_2(x) = 0 + g(x) \cdot 0 = 0.$$

Therefore $f_1 + gf_2 \in I_n$ and I_n is an ideal. Moreover if $f(x) = 0$ for all $x \geq n$, then $f(x) = 0$ for all $x \geq n + 1$, giving that $I_n \subseteq I_{n+1}$. Finally, for each n, consider the function $f_n : \mathbb{N} \to \mathbb{Z}/2\mathbb{Z}$ defined by $f_n(n) = 1$ and $f_n(m) = 0$ for all $m \neq n$. Then f_n belongs to I_{n+1} but not I_n, showing that $I_n \subsetneq I_{n+1}$. Therefore $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$ is an infinite ascending chain of ideals that does not terminate. This shows that the ring R is not Noetherian. □

Problem 2 (DF 15.2 Exercises 39,40)**.** Let R be a Noetherian ring and suppose

$$I = \cap_{i=1}^{m} Q_i$$

is a minimal primary decomposition of an ideal $I \subset R$. For each i, let $P_i = \text{rad}(Q_i)$ be the prime associated to Q_i. For $a \in R$, define

$$I : \langle a \rangle = \{r \in R : ar \in I\}.$$

(a) Show that $I : \langle a \rangle$ is an ideal of R and $I : \langle a \rangle = R$ if and only if $a \in I$.

Proof of (a). First, note that $0 = a \cdot 0 \in I$ and so $0 \in I : \langle a \rangle$. Suppose that $f, g \in I : \langle a \rangle$ and ler $r \in R$. Then af and ag belong to I, and therefore so does $af + rag = a(f + rg)$. This shows that $f + rg \in I : \langle a \rangle$, thus $I : \langle a \rangle$ is an ideal of R.

If $a \in I$, then for every $r \in R$, $ar \in I$, giving $R = I : \langle a \rangle$. Similarly, if $R = I : \langle a \rangle$, then $1 \in I : \langle a \rangle$, giving $a = a \cdot 1 \in I$. □

(b) Show that for any ideals I and J, $(I \cap J) : \langle a \rangle = (I : \langle a \rangle) \cap (J : \langle a \rangle)$.

Proof of (b). For $f \in R$, we have the following string of equivalent conditions:

$$\begin{aligned} f \in (I \cap J) : \langle a \rangle \quad &\Leftrightarrow \quad af \in I \cap J \\ &\Leftrightarrow \quad af \in I \text{ and } af \in J \\ &\Leftrightarrow \quad f \in I : \langle a \rangle \text{ and } f \in J : \langle a \rangle \\ &\Leftrightarrow \quad f \in (I : \langle a \rangle) \cap (J : \langle a \rangle) \end{aligned}$$

□

(c) Show that if $a \notin Q_i$, then $Q_i : \langle a \rangle$ is primary with $\text{rad}(Q_i : \langle a \rangle) = P_i$ and that if $a \notin P_i$, then $Q_i : \langle a \rangle = Q_i$.

Proof of (c). Suppose that $a \notin Q_i$ and suppose that $rs \in Q_i : \langle a \rangle$ with $s \notin Q_i : \langle a \rangle$. Then $a \cdot rs = r \cdot (as) \in Q_i$ and $as \notin Q_i$. Since Q_i is primary, it follows that $r^k \in Q_i \subseteq Q_i : \langle a \rangle$ for some $k \in \mathbb{Z}_+$. Therefore $Q_i : \langle a \rangle$ is primary. Since Q_i is a subset of $Q_i : \langle a \rangle$, $P_i = \text{rad}(Q_i)$ is contained in $\text{rad}(Q_i : \langle a \rangle)$. If $r^k \in Q_i : \langle a \rangle$, then $ar^k \in Q_i$. Since $a \notin Q_i$, some power $(r^k)^\ell = r^{k\ell}$ belongs to Q_i, showing that $r \in \text{rad}(Q_i)$. Therefore $\text{rad}(Q_i : \langle a \rangle) = \text{rad}(Q_i) = P_i$.

Suppose that $Q_i : \langle a \rangle \not\subset Q_i$. Then there exists some $r \in Q_i : \langle a \rangle$ with $r \notin Q_i$. By definition, $ra \in Q_i$, and since Q_i is primary and $r \notin Q_i$, $a^k \in Q_i$ for some k, giving that $a \in P_i$. This shows that if $a \notin P_i$, then $Q_i : \langle a \rangle$ is a subset of Q_i. Since $Q_i \subseteq Q_i : \langle a \rangle$ holds by definition, we see that they must be equal. □

(d) Show that

$$I : \langle a \rangle = \bigcap_{i=1}^{m} (Q_i : \langle a \rangle) \text{ and } \text{rad}(I : \langle a \rangle) = \bigcap_{i=1}^{m} \text{rad}(Q_i : \langle a \rangle).$$

Proof of (d). We can show the left equality by induction on m. For $m = 1$ this is clear. For $m > 1$, we write $I = J \cap Q_m$ where $J = \cap_{i=1}^{m-1} Q_i$. By part (b), we have

$$I : \langle a \rangle = (J : \langle a \rangle) \cap (Q_m : \langle a \rangle).$$

By induction $J : \langle a \rangle = \cap_{i=1}^{m-1} (Q_i : \langle a \rangle)$, giving that $I : \langle a \rangle = \cap_{i=1}^{m} (Q_i : \langle a \rangle)$.

The statement that $\text{rad}(I : \langle a \rangle) = \cap_{i=1}^{m} \text{rad}(Q_i : \langle a \rangle)$ then follows from the following:

Lemma 1. *For any collection of ideals $J_1, \ldots, J_m \subseteq R$, $\text{rad}(\cap_{i=1}^{m} J_i) = \cap_{i=1}^{m} \text{rad}(J_i)$.*

Proof of Lemma. Let $f \in R$. Then

$$\begin{aligned} f \in \text{rad}(\cap_{i=1}^{m} J_i) &\Leftrightarrow f^k \in \cap_{i=1}^{m} J_i \text{ for some } k \in \mathbb{N} \\ &\Leftrightarrow f^k \in J_i \text{ for some } k \in \mathbb{N} \text{ and for every } i = 1, \ldots, m \\ \text{(take } k = \max_i k_i \text{ or } k_i = k) \qquad &\Leftrightarrow f^{k_i} \in J_i \text{ for every } i = 1, \ldots, m \text{ and for some } k_i \in \mathbb{N} \\ &\Leftrightarrow f \in \text{rad}(J_i) \text{ for every } i = 1, \ldots, m \\ &\Leftrightarrow f \in \cap_{i=1}^{m} \text{rad}(J_i). \end{aligned}$$

□

□

(e) Show that $\mathrm{rad}(I : \langle a\rangle)$ is the intersection of the primes P_i for which $a \notin Q_i$.

Proof of part (e). By part (d),

$$\mathrm{rad}(I : \langle a\rangle) = \bigcap_{i=1}^{m} \mathrm{rad}(Q_i : \langle a\rangle).$$

Note that by part (a), if $a \in Q_i$, then $Q_i : \langle a\rangle = R$ and $\mathrm{rad}(Q_i : \langle a\rangle) = R$, meaning that it does not contribute to the intersection above. If $a \notin Q_i$, then by part (c), $\mathrm{rad}(Q_i : \langle a\rangle) = P_i$. Therefore

$$\mathrm{rad}(I : \langle a\rangle) = \bigcap_{i:a\notin Q_i} \mathrm{rad}(Q_i : \langle a\rangle) = \bigcap_{i:a\notin Q_i} P_i.$$

□

(f) Show that if $\mathrm{rad}(I : \langle a\rangle)$ is prime, then $\mathrm{rad}(I : \langle a\rangle) = P_i$ for some i.

Proof of (f). By part (e), $\mathrm{rad}(I : \langle a\rangle) = \cap_{i:a\notin Q_i} P_i$ is an intersection of prime ideals. If $\mathrm{rad}(I : \langle a)$ is prime, then by the lemma below, it must be equal to one of those primes.

Lemma 2. *If $P_1, \dots, P_m \subset R$ are prime and $\cap_{i=1}^m P_i$ is prime, then $\cap_{i=1}^m P_i = P_j$ for some j.*

Proof of Lemma. Let $P_1, \dots, P_m \subset R$ be prime ideals. Suppose that for every $j = 1, \dots, m$, $\cap_{i=1}^m P_i \neq P_j$. Since the intersection is contained in P_j, this implies $\cap_{i=1}^m P_i \subsetneq P_j$. For each $j = 1, \dots, m$, let a_j be an element of P_j that is not in $\cap_{i=1}^m P_i$. Then $\prod_{i=1}^m a_i$ is belongs to the intersection $\cap_{i=1}^m P_i$, but none of its factors a_j do. This shows that $\cap_{i=1}^m P_i$ is prime. The contrapositive gives the lemma. □

□

(g) Show that for each $i = 1, \dots, m$, there exists an element $a \in R$ with $\mathrm{rad}(I : \langle a\rangle) = P_i$. (Hint: consider $a \in (\cap_{j\neq i} Q_j)\backslash Q_i$.)

Proof of (g). Fix $i \in [m]$. Since $\cap_j Q_j$ is a *minimal* primary decomposition, $\cap_{j\neq i} Q_j$ is not a subset of Q_i, meaning that there exists an element a in $\cap_{j\neq i} Q_j$ with $a \notin Q_i$. By part (e),

$$\mathrm{rad}(I : \langle a\rangle) = \bigcap_{j:a\notin Q_j} P_j = P_i.$$

The last equality holds because $a \in Q_j$ for all $j \neq i$. □

(h) Show that for a prime ideal $P \subseteq R$, P has the form $\mathrm{rad}(I : \langle a\rangle)$ for some $a \in R$ if and only if $P = P_i$ for some i.

Proof of (h). Let $P \subset R$ be a prime ideal.

($\Rightarrow$) If P has the form $\mathrm{rad}(I : \langle a\rangle)$ for some $a \in R$, then $\mathrm{rad}(I : \langle a\rangle)$ is prime and by part (f), $P = \mathrm{rad}(I : \langle a\rangle) = P_i$ for some i.

($\Leftarrow$) If $P = P_i$ for some i, then by part (g), there exists an element $a \in R$ for which $\mathrm{rad}(I : \langle a\rangle) = P$. □

This shows that the set of primes associated to an ideal I is unique.