Math 721 – Homework 7

Due Friday, March 6 at 5pm

Good practice problems (do not turn in solutions):
DF 15.1: 1, 6, 7, 8 10, 11, 29, 30
DF 15.2: 2, 7, 30, 31, 32, 34, 38

Problem 1 (DF 15.1 Exercise 2)**.** Show that each of the following rings are *not* Noetherian by exhibiting an explicit infinite increasing chain of ideals.

(a) the ring of continuous real valued functions on $[0, 1]$,

(b) the ring of all functions from $\mathbb{N}$ to $\mathbb{Z}/2\mathbb{Z}$.

Problem 2 (DF 15.2 Exercises 39,40)**.** Let R be a Noetherian ring and suppose

$$I = \cap_{i=1}^{m} Q_i$$

is a minimal primary decomposition of an ideal $I \subset R$. For each i, let $P_i = \mathrm{rad}(Q_i)$ be the prime associated to Q_i. For $a \in R$, define

$$I : \langle a \rangle = \{r \in R : ar \in I\}.$$

(a) Show that $I : \langle a \rangle$ is an ideal of R and $I : \langle a \rangle = R$ if and only if $a \in I$.

(b) Show that for any ideals I and J, $(I \cap J) : \langle a \rangle = (I : \langle a \rangle) \cap (J : \langle a \rangle)$.

(c) Show that if $a \notin Q_i$, then $Q_i : \langle a \rangle$ is primary with $\mathrm{rad}(Q_i : \langle a \rangle) = P_i$ and that if $a \notin P_i$, then $Q_i : \langle a \rangle = Q_i$.

(d) Show that

$$I : \langle a \rangle = \cap_{i=1}^{m} (Q_i : \langle a \rangle) \text{ and } \mathrm{rad}(I : \langle a \rangle) = \cap_{i=1}^{m} \mathrm{rad}(Q_i : \langle a \rangle).$$

(e) Show that $\mathrm{rad}(I : \langle a \rangle)$ is the intersection of the primes P_i for which $a \notin Q_i$.

(f) Show that if $\mathrm{rad}(I : \langle a \rangle)$ is prime, then $\mathrm{rad}(I : \langle a \rangle) = P_i$ for some i.

(g) Show that for each $i = 1, \dots, m$, there exists an element $a \in R$ with $\mathrm{rad}(I : \langle a \rangle) = P_i$. (Hint: consider $a \in (\cap_{j \neq i} Q_j) \backslash Q_i$.)

(h) Show that for a prime ideal $P \subseteq R$, P has the form $\mathrm{rad}(I : \langle a \rangle)$ for some $a \in R$ if and only if $P = P_i$ for some i.

This shows that the set of primes associated to an ideal I is unique.