Math 721 – Homework 6 Solutions

Problem 1 (DF 12.3 Exercise 19)**.** Prove that all $n \times n$ matrices over F with characteristic polynomial $f(x)$ are similar if and only if $f(x)$ has no repeated factors in its unique factorization in $F[x]$.

Proof. ($\Rightarrow$) Suppose that f has a repeated factor in its unique factorization in $F[x]$, i.e. $f = f_1 \cdot f_2 \cdots f_k$ and $f_1 = f_2$, where each $f_i \in F[x]$ has degree ≥ 1. Without loss of generality, we can take f and all its factors f_i to be monic. Let $g = f_1$ and $h = f_3 \cdots f_k$, so that $f = g^2 h$. Consider the matrices

$$A = \mathcal{C}_f \quad \text{and} \quad B = \begin{pmatrix} \mathcal{C}_g & 0 \\ 0 & \mathcal{C}_{g \cdot h} \end{pmatrix}$$

where $\mathcal{C}_a$ denotes the companion matrix of the polynomial a. Since g divides $g \cdot h$, both of these matrices are in rational canonical form and have characteristic polynomial $f = g \cdot g \cdot h$. Since their rational canonical forms are different, they are not similar.

($\Leftarrow$) Suppose that $f(x)$ has no repeated factors in its unique factorization in $F[x]$. We know that the companion matrix $\mathcal{C}_f$ has characteristic polynomial f. Suppose that A is any matrix with characteristic polynomial f, and let $a_1(x), \ldots, a_m(x) \in F[x]$ denote the invariant factors of A. Note that the characteristic polynomial of f is $f = a_1 \cdots a_m$. If $m > 1$, then a_1 divides a_2. If f_1 is an irreducible factor of a_1 in $F[x]$, then f_1^2 divides f, meaning that f has a repeated factors in its unique factorization, giving a contradiction. Therefore $m = 1$ and $a_1 = f$. The rational canonical form of A is therefore $\mathcal{C}_f$. □

Problem 2 (DF 12.3 Exercises 29, 30)**.** Let V be a vectorspace over a field F and $T : V \to V$ a linear transformation whose eigenvalues all lie in F. For any eigenvalue λ of T, the *generalized eigenspace* of T corresponding to λ is the p-primary component of V as a $F[x]$-module corresponding to the prime $p = x - \lambda$. Equivalently, it is the subspace of vectors annihilated by some power of the linear operator $T - \lambda \cdot \mathrm{id}_V$.

Let λ be an eigenvalue T and let W denote the corresponding generalized eigenspace. Suppose that V is finite dimensional.

(a) Show that for any $k \geq 0$ the dimension of the kernel of $T - \lambda \cdot \mathrm{id}$ on the vectorspace $(T - \lambda \cdot \mathrm{id})^k W$ equals the dimension of the kernel of $T - \lambda \cdot \mathrm{id}$ on the vectorspace $(T - \lambda \cdot \mathrm{id})^k V$, and that this equals the number of Jordan blocks of T having eigenvalue λ and size $> k$.

(b) Let $r_k = \dim_F (T - \lambda \cdot \mathrm{id})^k V$. Show that for any $k \geq 1$, the number of Jordan blocks of size k with eigenvalue λ equals $r_{k-1} - 2r_k + r_{k+1}$. (You may use DF 12.1 Exercise 12 without proof.)

Proof of (a). Consider V as an $F[x]$-module defined by $x \cdot v = T(v)$ for $v \in V$. Then for any $\mu \in F$,

$$(x - \mu)^k \cdot v = (T - \mu \cdot \mathrm{id})^k (v).$$

Let $(x - \lambda_1)^{\alpha_1}, \ldots, (x - \lambda_s)^{\alpha_s}$ denote the list of elementary divisors of T, so that as $F[x]$-modules we have

$$V \cong F[x]/\langle (x - \lambda_1)^{\alpha_1} \rangle \oplus \ldots \oplus F[x]/\langle (x - \lambda_s)^{\alpha_s} \rangle.$$

For simplicity of notation, we will take the above isomorphism to be an equality in the rest of the proof. For any $f \in F[x]$, the linear transformation given by multiplication by f is done component-wise. Note that if $\lambda \neq \lambda_i$, $x-\lambda$ and $(x-\lambda_i)^{\alpha_i}$ are relatively prime and $x-\lambda$ is a unit in $F[x]/\langle(x-\lambda_i)^{\alpha_i}$. In particular, multiplication by $x-\lambda$ gives an invertible linear transformation on $F[x]/\langle(x-\lambda_i)^{\alpha_i}\rangle$.

Moreover, for $\lambda = \lambda_i$, the polynomial $(x-\lambda)^{\alpha_i}$ acts as the zero map on $F[x]/\langle(x-\lambda_i)^{\alpha_i}$. Therefore, under the isomorphism above,

$$W = \bigoplus_{\{i:\lambda_i=\lambda\}} F[x]/\langle(x-\lambda_i)^{\alpha_i}\rangle.$$

Then

$$(T-\lambda\mathrm{id})^k V = \bigoplus_{i=1}^{s}(x-\lambda)^k F[x]/\langle(x-\lambda_i)^{\alpha_i}\rangle \quad \text{and} \quad (T-\lambda\mathrm{id})^k W = \bigoplus_{\{i:\lambda_i=\lambda\}} (x-\lambda)^k F[x]/\langle(x-\lambda_i)^{\alpha_i}\rangle.$$

Suppose that $v = ((x-\lambda)^k f_i + \langle(x-\lambda_i)^{\alpha_i}\rangle)_{i=1}^{s}$ in $(T-\lambda\mathrm{id})^k V$ belongs to the kernel of $T-\lambda\mathrm{id}$. This implies that $(x-\lambda)\cdot(x-\lambda)^k f_i \in \langle(x-\lambda_i)^{\alpha_i}\rangle$ for all $i = 1,\ldots,s$. When $\lambda \neq \lambda_i$, $(x-\lambda)^{k+1}$ and $(x-\lambda_i)^{\alpha_i}$ are relatively prime, and so this implies that $f_i \equiv 0$ modulo $\langle(x-\lambda_i)^{\alpha_i}\rangle$. This implies that $v \in (T-\lambda\mathrm{id})^k W$. Therefore the kernel of $T-\lambda\mathrm{id}$ on $(T-\lambda\mathrm{id})^k V$ equals its kernel on $(T-\lambda\mathrm{id})^k W$.

Moreover, for $\lambda = \lambda_i$, the element $(x-\lambda)^k f_i + \langle(x-\lambda)^{\alpha_i}\rangle$ is non-zero only when $k < \alpha_i$ and $f_i \notin \langle(x-\lambda)^{\alpha_i-k}\rangle$. For $k \geq \alpha_i$, $(x-\lambda)^k F[x]/\langle(x-\lambda)^{\alpha_i}\rangle = \{0\}$. Suppose that $k < \alpha_i$. This belongs to the kernel of the map given by multiplication by $x-\lambda$ if and only if $(x-\lambda)^{k+1} f_i \in \langle(x-\lambda)^{\alpha_i}\rangle$, which occurs if and only if $f_i \in \langle(x-\lambda)^{\alpha_i-k-1}\rangle$. This shows that the kernel of multiplication by $(x-\lambda)$ on the space $(x-\lambda)^k F[x]/\langle(x-\lambda)^{\alpha_i}\rangle$ is the one dimensional subspace spanned by $(x-\lambda)^{\alpha_i-1}$.

Put together, this shows that $\{(x-\lambda)^{\alpha_i-1}e_i : \lambda = \lambda_i,\ k < \alpha_i\}$ is a basis for the kernel of multiplication by $(x-\lambda)$ on

$$(T-\lambda\cdot\mathrm{id})^k W = \bigoplus_{\{i:\lambda_i=\lambda\}} (x-\lambda)^k F[x]/\langle(x-\lambda_i)^{\alpha_i}\rangle.$$

Here, for each $i = 1,\ldots,s$, let e_i denote the element of $\bigoplus_{i=1}^{s}(x-\lambda)^k F[x]/\langle(x-\lambda_i)^{\alpha_i}\rangle$ with a 1 in the ith coordinate and zero elsewhere. In particular, the dimension of this kernel is $|\{i : \lambda = \lambda_i,\ k < \alpha_i\}|$, which is the number of Jordan blocks of T with eigenvalues λ and size $> k$. □

Proof of (b). Let $r_k = \dim_F(T-\lambda\cdot\mathrm{id})^k V$ and let d_k denote the dimension of the kernel of $T-\lambda\mathrm{id}$ on this vectorspace (as in part (a)). Note that the image of $(T-\lambda\cdot\mathrm{id})^k V$ under the map $T-\lambda\cdot\mathrm{id}$ is $(T-\lambda\cdot\mathrm{id})^{k+1}V$, which has dimension r_{k+1}. Since the image is isomorphic to the quotient of the domain by the kernel, we find that

$$r_{k+1} = r_k - d_k \quad \text{and thus} \quad d_k = r_k - r_{k+1}.$$

By part (a), d_k equals the number of Jordan blocks with eigenvalue λ and size $> k$. The number of Jordan blocks with eigenvalue λ and size $= k$ is the different between the number with size $> k-1$ and the number with size $> k$, which is then

$$d_{k-1} - d_k = (r_{k-1} - r_k) - (r_k - r_{k+1}) = r_{k-1} - 2r_k + r_{k+1}.$$

□