Good practice problems (do not turn in solutions):
DF 12.3 Exercises 1, 2, 17, 18, 22, 23, 24, 32, 33, 37

**Problem 1** (DF 12.3 Exercise 19). Prove that all \( n \times n \) matrices over \( F \) with characteristic polynomial \( f(x) \) are similar if and only if \( f(x) \) has no repeated factors in its unique factorization in \( F[x] \).

**Problem 2** (DF 12.3 Exercises 29, 30). Let \( V \) be a vectorspace over a field \( F \) and \( T : V \to V \) a linear transformation whose eigenvalues all lie in \( F \). For any eigenvalue \( \lambda \) of \( T \), the **generalized eigenspace** of \( T \) corresponding to \( \lambda \) is the \( p \)-primary component of \( V \) as a \( F[x] \)-module corresponding to the prime \( p = x - \lambda \). Equivalently, it is the subspace of vectors annihilated by some power of the linear operator \( T - \lambda \cdot \text{id}_V \).

Let \( \lambda \) be an eigenvalue of \( T \) and let \( W \) denote the corresponding generalized eigenspace. Suppose that \( V \) is finite dimensional.

(a) Show that for any \( k \geq 0 \) the dimension of the kernel of \( T - \lambda \cdot \text{id} \) on the vectorspace \((T - \lambda \cdot \text{id})^kW\) equals the dimension of the kernel of \( T - \lambda \cdot \text{id} \) on the vectorspace \((T - \lambda \cdot \text{id})^kV\), and that this equals the number of Jordan blocks of \( T \) having eigenvalue \( \lambda \) and size > \( k \).

(b) Let \( r_k = \dim_F(T - \lambda \cdot \text{id})^kV \). Show that for any \( k \geq 1 \), the number of Jordan blocks of size \( k \) with eigenvalue \( \lambda \) equals \( r_{k-1} - 2r_k + r_{k+1} \). (You may use DF 12.1 Exercise 12 without proof.)