Math 721 – Homework 5 Solutions

Problem 1 (DF 12.1 Exercise 6)**.** Show that if R is an integral domain and M is any non-principal ideal of R, then as an R-module, M is torsion-free of rank 1, but is not a free R-module.

Proof. Let R be an integral domain and let M be a non-principal ideal of R. To see that M is torsion-free, let $r \in R \backslash \{0\}$ and $m \in M$. Since R is an integral domain, $rm = 0$ implies that $m = 0$, giving that $\mathrm{Tor}(M) = \{0\}$ and M is torsion free.

Since M is non-principal, $M \neq \{0\}$ and so it has rank ≥ 1. Any two nonzero elements $a, b \in M\backslash\{0\}$ satisfy an R-linear relation, namely

$$(-b) \cdot a + (a) \cdot b = 0,$$

and so are not R-linearly independent. Therefore the rank of M at most one.

To see that M is not a free R-module, suppose for the sake of contradiction that M is a free R-module of rank n. Since the rank of M is one, we see immediately that $n = 1$. Thus there is some element $a \in M$ for which $M = R \cdot a$. This means that M is the principal ideal $\langle a \rangle$, contradicting our assumption that M is not principal. Therefore M is not free. □

Problem 2 (DF 12.2 Exercises 3, 4)**.** Let F be a field.

(a) Prove that two 2×2 matrices over F which are not scalar multiple of the identity matrix are similar if and only if they have the same characteristic polynomial.

(b) Prove that two 3×3 matrices over F are similar if and only if they have the same characteristic and minimal polynomials.

(c) Give an explicit counterexample to the statement of (b) for 4×4 matrices.

First, note that any two similar matrices have the same invariant factors $a_1(x), \ldots, a_m(x)$ and therefore the same characteristic polynomial $\prod_{i=1}^m a_i(x)$ and minimal polynomial $a_m(x)$. This gives ($\Rightarrow$) in parts (a) and (b).

Proof of (a). Suppose that $A, B \in \mathrm{Mat}_2(F)$ have characteristic polynomials $c_A(x) = c_B(x)$. Let $a_1(x), \ldots, a_m(x)$ be the invariant factors of A.

If $m > 1$, then $m = 2$. Moreover, since $a_1(x)$ divides $a_2(x)$ and their product is the degree-two polynomial $c_A(x)$, both must must linear and equal. Then $a_1(x) = a_2(x) = x - \lambda$ for some $\lambda \in F$. These are exactly the invariant factors of $\lambda \cdot \mathrm{id}_2$. Note that the only matrix similar to $\lambda \cdot \mathrm{id}_2$ is itself, therefore this would imply that $A = \lambda \cdot \mathrm{id}_2$. Since A is not a scalar multiple of the identity matrix, it must be that $m = 1$ and $a_1(x) = c_A(x)$.

Similarly, B can only have one invariant factor, which must equal $c_B(x)$. Since A and B have the same list of invariant factors, they must be similar. □

Proof of (b). Suppose that $A, B \in \mathrm{Mat}_3(F)$ have minimal polynomials $m_A(x) = m_B(x)$ and characteristic polynomials $c_A(x) = c_B(x)$. Let $a_1(x), \ldots, a_m(x)$ be the invariant factors of A.

If the degree of $m_A(x)$ is one, then $a_m = m_A(x) = x - \lambda$ for some $\lambda \in F$. Since the invariant factors must divide each other and multiply to the cubic polynomial $c_A(x)$, then gives that $m = 3$ and $a_1(x) = a_2(x) = a_3(x) = x - \lambda$. The same argument holds for the invariant factors of B. Since A and B have the same list of invariant factors, they must be similar.

If the degree of $m_A(x)$ is two, then $m = 2$ and $a_2(x) = m_A(x)$. Then $a_1(x) = c_A(x)/m_A(x)$. Again, the list of invariant factors of A is uniquely determined by $m_A(x)$ and $c_A(x)$, so B must have the same list of invariant factors and be similar.

If the degree of $m_A(x)$ is three, then $m = 1$ and $a_1(x) = c_A(x)$. Since the list of invariant factors of A is uniquely determined by $m_A(x)$ and $c_A(x)$, B must have the same list of invariant factors and be similar. □

Proof of (c). Consider the lists of invariant factors $a_1 = x$, $a_2 = x$, $a_3 = x^2$ and $b_1 = x^2$, $b_2 = x^2$ with corresponding matrices

$$A = \begin{pmatrix} C_{a_1} & & \\ & C_{a_2} & \\ & & C_{a_3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} C_{b_1} & \\ & C_{b_2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

These have the same minimal polynomial $a_3 = b_2 = x^2$ and same characteristic polynomials $a_1 \cdot a_2 \cdot a_3 = b_1 \cdot b_2 = x^4$. Since they do not have the same lists of invariant factors, they are not similar. □