# Math 721 – Homework 4 Solutions

**Problem 1** (Graded rings and quotients)**.**

(a) [DF Exercise 11.5.2] Fill in the details for the proof of Proposition 11.5.33. That is, show that for a graded ideal $I$ of a graded ring $S$ with $I_k = I \cap S_k$, the quotient ring $S/I$ is a graded ring with $(S/I)_k \cong S_k/I_k$. (Hint: it may be useful to check out the hint in Exercise 2 and the proof sketch of Proposition 33 of DF §11.5.)

(b) Consider the graded ring $S = \mathbb{Q}[x, y]$ and ideal $I = \langle x^4, y^4 \rangle$. Give a basis for each of the following $\mathbb{Q}$-vectorspaces (and justify your answers):

(i) $S_5$,

(ii) $I_5$,

(iii) $(S/I)_5$.

*Proof of (a).* Let $S = \oplus_{k=0}^{\infty} S_k$ be a graded ring and $I = \oplus_{k=0}^{\infty} I_k$ be a graded ideal of $S$, with $I_k = S_k \cap I$. First, we show that $\oplus_{k=0}^{\infty} S_k/I_k$ is a graded ring under coordinate-wise addition and multiplication defined by

$$(s_i + I_i)_i \cdot (s'_j + I_j)_j = (\sum_{i+j=k} s_i s'_j + I_k)_k.$$

In particular, we need to show that this multiplication is well defined. Suppose $r_i + I_i = s_i + I_i$ and $r'_j + I_j = s'_j + I_j$. Then $(r_i - s_i)r'_j$ and $s_i(r'_j - s'_j)$ both belong to $S_{i+j} \cap I = I_{i+j}$. Note that we can write

$$r_i r'_j = s_i s'_j + (r_i - s_i)r'_j + s_i(r'_j - s'_j),$$

which shows that

$$r_i r'_j + I_{i+j} = s_i s'_j + I_{i+j}.$$

Thus the multiplication on $\oplus_{k=0}^{\infty} S_k/I_k$ is well-defined.

To see that $R = \oplus_{k=0}^{\infty} S_k/I_k$ is a ring with the multiplication, note first that it is immediate that $R$ is an abelian group under coordinate-wise addition. To check that multiplication is associative, note that

$$((a_i + I_i)_i \cdot (b_j + I_j)_j) \cdot (c_k + I_k)_k = (\sum_{i+j+k=\ell} a_i b_j c_k + I_\ell)_\ell = (a_i + I_i)_i \cdot ((b_j + I_j)_j \cdot (c_k + I_k)_k).$$

For distributivity,

$$\begin{aligned} ((a_i + I_i)_i + (b_i + I_i)_i) \cdot (c_j + I_j)_j &= (a_i + b_i + I_i)_i \cdot (c_j + I_j)_j \\ &= (\sum_{i+j=k} (a_i + b_i)c_j + I_k)_k \\ &= (\sum_{i+j=k} a_i c_j + I_k)_k + (\sum_{i+j=k} b_i c_j + I_k)_k \\ &= ((a_i + I_i)_i \cdot (c_j + I_j)_j) + ((b_i + I_i)_i \cdot (c_j + I_j)_j). \end{aligned}$$

Therefore $\oplus_{k=0}^{\infty} S_k/I_k$ is a ring. It follows immediately from the definition of multiplication that it is graded, i.e. $(S_i/I_i) \cdot (S_j/I_j) \subseteq S_{i+j}/I_{i+j}$.

Now consider the homomorphism $\pi : S \to \oplus_{k=0}^{\infty} S_k/I_k$ given by

$$\pi\big((s_k)_k\big) = (s_k + I_k)_k$$

Then

$$\begin{aligned}\pi\big((a_k)_k + (b_k)_k\big) &= \pi\big((a_k + b_k)_k\big)\\ &= (a_k + b_k + I_k)_k\\ &= (a_k + I_k)_k + (b_k + I_k)_k\\ &= \pi\big((a_k)_k\big)+\pi\big((b_k)_k\big)\end{aligned}$$

and

$$\begin{aligned}\pi\big((a_i)_i \cdot (b_j)_j\big) &= \pi\big((\sum_{i+j=k} a_i b_j)_k\big)\\ &= (\sum_{i+j=k} a_i b_j + I_k)_k\\ &= \pi\big((a_i)_i\big)\cdot\pi\big((b_j)_j\big)\end{aligned}$$

Note that $\pi$ is surjective, since any element $(s_k + I_k)_k \in \oplus_{k=0}^{\infty} S_k/I_k$ is the image of $(s_k)_k \in \oplus_{k=0}^{\infty} S_k = S$ under $\pi$.

Finally, note that $(s_k)_k$ belongs to the kernel of $\pi$ if and only if

$$(s_k + I_k)_k = (0 + I_k)_k.$$

This occurs if and only if $s_k \in I_k$ for all $k$, or equivalently $(s_k)_k \in \oplus_{k=0}^{\infty} I_k = I$. Therefore $\ker(\pi) = I$.

The first isomorphism theorem that $S/\ker(\pi) \cong \text{image}(\pi)$ then shows that

$$S/I \cong \oplus_{k=0}^{\infty} S_k/I_k.$$

□

*Proof of (b).* Consider the graded ring $S = \mathbb{Q}[x, y]$ and ideal $I = \langle x^4, y^4\rangle$. Note that the monomials of degree $d$ form a basis for $S_d$, so

$$\{x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5\} \text{ is a basis for } S_5.$$

We claim that

$$\{x^5, x^4y, xy^4, y^5\} \text{ is a basis for } I_5 = S_5 \cap I.$$

Note that each of these elements is a multiple of $x^4$ or $y^4$, showing that they belong to $I_5$. Moreover, their linear independence in $S_5$ implies their linear independence $I_5$. Elements of $I$ have the form $x^4f + y^4g$. In particular, expanding the monomial expressions for $f$ and $g$ show that $x^4f + y^4g$ is a linear combination of monomials of the form $x^ay^b$ with $a \geq 4$ or $b \geq 4$. Therefore the set above also spans $I_5 = S_5 \cap I$.

By the argument above, no polynomial of the form $c_1x^3y^2 + c_2x^2y^3$ belongs to $I$, meaning that $x^3y^2 + I_5$ and $x^2y^3 + I_5$ are linearly independent in $S_5/I_5$. Every element of $S_5$ is equivalent to a polynomial of this form, modulo $I$, which means that $x^3y^2 + I_5$ and $x^2y^3 + I_5$ also spans $S_5/I_5$. Therefore

$$\{x^3y^2 + I_5, x^2y^3 + I_5\} \text{ is a basis for } S_5/I_5.$$

□

**Problem 2.** Let $F$ be any field of characteristic $\mathrm{char}(F) \neq 2$ (so that $-1 \neq 1$). Let $V$ be any vectorspace over $F$.

(a) [DF Exercise 11.5.13] Prove that $V \otimes_F V = \mathcal{S}^2(V) \oplus \bigwedge^2(V)$, i.e. that every 2-tensor may be written uniquely as a sum of a symmetric and an alternating tensor.

(b) Supposing that $V$ is an $n$-dimensional vectorspace over $F$, show the following isomorphisms of $F$-modules by producing (and checking) explicit isomorphisms:

(i) $V \otimes_F V \cong \mathrm{Mat}_n(F)$,

(ii) $\mathcal{S}^2(V) \cong \mathrm{Sym}_n(F)$ where $\mathrm{Sym}_n(F) = \{A \in \mathrm{Mat}_n(F) : A = A^T\}$,

(iii) $\bigwedge^2(V) \cong \mathrm{Skew}_n(F)$ where $\mathrm{Skew}_n(F) = \{A \in \mathrm{Mat}_n(F) : A = -A^T\}$.

*Proof of (b).* Consider the map $V \times V \mapsto \{\text{symmetric 2-tensors}\} \oplus \{\text{antisymmetric 2-tensors}\}$ given by

$$(v, w) \mapsto \left(\frac{1}{2}(v \otimes w + w \otimes v), \frac{1}{2}(v \otimes w - w \otimes v)\right)$$

It is straightforward to check that it is bilinear and extends to an $F$-linear map $\Phi : V \otimes_F V \to \{\text{symmetric 2-tensors}\} \oplus \{\text{antisymmetric 2-tensors}\}$ by

$$\sum_i v_i \otimes w_i \mapsto \left(\sum_i \frac{1}{2}(v_i \otimes w_i + w_i \otimes v_i), \sum_i \frac{1}{2}(v_i \otimes w_i - w_i \otimes v_i)\right)$$

Consider the $F$-linear map $\Psi : \{\text{symmetric 2-tensors}\} \oplus \{\text{antisymmetric 2-tensors}\} \to V \otimes_F V$ given by $\Psi(s, a) = s + a$.

Note that

$$\Psi(\Phi(\sum_i v_i \otimes w_i)) = \Psi\left(\sum_i \frac{1}{2}(v_i \otimes w_i + w_i \otimes v_i), \sum_i \frac{1}{2}(v_i \otimes w_i - w_i \otimes v_i)\right) = \sum_i v_i \otimes w_i$$

and so $\Psi \circ \Phi = \mathrm{id}_{V \otimes_F V}$.

Note that for any symmetric tensor $s = \sum_i v_i \otimes w_i$

$$\sum_i v_i \otimes w_i = s = \sigma s = \sum_i w_i \otimes v_i$$

where $\sigma = (12)$ and so

$$\Phi(s) = \left(\frac{1}{2}(s + \sigma s), \frac{1}{2}(s - \sigma s)\right) = (s, 0).$$

Similarly, for any antisymmetric tensor $a = \sum_i v_i \otimes w_i$

$$\sum_i v_i \otimes w_i = a = -\sigma a = -\sum_i w_i \otimes v_i$$

and so

$$\Phi(a) = \left(\frac{1}{2}(a + \sigma a), \frac{1}{2}(a - \sigma a)\right) = (0, a).$$

Then for any symmetric tensor $s$ and antisymmetric tensor $a$,

$$\Phi(\Psi(s, a)) = \Phi(s + a) = \Phi(s) + \Phi(a) = (s, 0) + (0, a) = (s, a).$$

Therefore $\Phi \circ \Psi = \mathrm{id}$.

This shows that $V \otimes_F V$ is isomorphism to the direct product $\{\text{symmetric 2-tensors}\} \oplus \{\text{antisymmetric 2-tensors}\}$

By Proposition 11.5.40 in the book, the maps $\mathcal{S}^2(M)$ is isomorphic to the submodule $\{\text{symmetric 2-tensors}\}$ and $\bigwedge^2(M)$ is isomorphic to the submodule $\{\text{antisymmetric 2-tensors}\}$. Therefore $V \otimes_F V$ is isomorphic to $\mathcal{S}^2(M) \oplus \bigwedge^2(M)$. □

*Proof of (b).* Suppose that $V$ is an $n$-dimensional vector space over $F$ with basis $\{v_1, \dots, v_n\}$. Then $\{v_i \otimes v_j : 1 \leq i, j \leq n\}$ is a basis for $V \otimes_F V$. The matrices $\{E_{ij} : 1 \leq i, j \leq n\}$ is a basis for the space of $n \times n$ matrices where $E_{ij}$ is a matrix with 1 in $i$th row and $j$th column and 0's elsewhere. The linear map $L : V \otimes_F V \to \text{Mat}_n(F)$ determined by $L(v_i \otimes v_j) = E_{ij}$ is an isomorphism.

Note that for $\sigma = (12)$ and $t \in V \otimes_F V$, $L(\sigma t) = L(t)^T$ where $A^T$ denotes the transpose of $A \in \text{Mat}_n(F)$. In particular, $L(t) = L(t)^T = L(\sigma t)$ if and only if $t = \sigma t$. Therefore $L(t)$ is a symmetric matrix if and only if the tensor $t \in V \otimes_F V$ is symmetric. Therefore $L$ restricts to an isomorphism between $\{\text{symmetric 2-tensors}\}$ and $\text{Sym}_n(F)$. By Proposition 11.5.40, $\mathcal{S}^2(V) \cong \{\text{symmetric 2-tensors}\}$, so $\mathcal{S}^2(V) \cong \text{Sym}_n(F)$.

Similarly for $t \in V \otimes_F V$, $L(t) = -L(t)^T = -L(\sigma t)$ if and only if $t = -\sigma t$. Therefore $L(t)$ is a skew symmetric matrix if and only if the tensor $t \in V \otimes_F V$ is antisymmetric. So $L$ restricts to an isomorphism between $\{\text{antisymmetric 2-tensors}\}$ and $\text{Skew}_n(F)$. By Proposition 11.5.40, $\bigwedge^2(V) \cong \{\text{antisymmetric 2-tensors}\}$, so $\bigwedge^2(V) \cong \text{Skew}_n(F)$. □