Math 721 – Homework 4

Due Friday, February 7 at 5pm

Good practice problems (do not turn in solutions):
DF 11.5 Exercises 1, 4, 5, 10

Problem 1 (Graded rings and quotients)**.**

(a) [DF Exercise 11.5.2] Fill in the details for the proof of Proposition 11.5.33. That is, show that for a graded ideal I of a graded ring S with $I_k = I \cap S_k$, the quotient ring S/I is a graded ring with $(S/I)_k \cong S_k/I_k$. (Hint: it may be useful to check out the hint in Exercise 2 and the proof sketch of Proposition 33 of DF §11.5.)

(b) Consider the graded ring $S = \mathbb{Q}[x, y]$ and ideal $I = \langle x^4, y^4 \rangle$. Give a basis for each of the following $\mathbb{Q}$-vectorspaces (and justify your answers):

(i) S_5,
(ii) I_5,
(iii) $(S/I)_5$.

Problem 2. Let F be any field of characteristic $\operatorname{char}(F) \neq 2$ (so that $-1 \neq 1$). Let V be any vectorspace over F.

(a) [DF Exercise 11.5.13] Prove that $V \otimes_F V = \mathcal{S}^2(V) \oplus \bigwedge^2(V)$, i.e. that every 2-tensor may be written uniquely as a sum of a symmetric and an alternating tensor.

(b) Supposing that V is an n-dimensional vectorspace over F, show the following isomorphisms of F-modules by producing (and checking) explicit isomorphisms:

(i) $V \otimes_F V \cong \mathrm{Mat}_n(F)$,
(ii) $\mathcal{S}^2(V) \cong \mathrm{Sym}_n(F)$ where $\mathrm{Sym}_n(F) = \{A \in \mathrm{Mat}_n(F) : A = A^T\}$,
(iii) $\bigwedge^2(V) \cong \mathrm{Skew}_n(F)$ where $\mathrm{Skew}_n(F) = \{A \in \mathrm{Mat}_n(F) : A = -A^T\}$.