Math 721 – Homework 3 Solutions

Problem 1 (DF Exercise 3, 4, 5)**.** Let X_1, X_2 be R-modules.

(a) Show that $X_1 \oplus X_2$ is projective if and only if X_1 and X_2 are projective.
(b) Show that $X_1 \oplus X_2$ is injective if and only if X_1 and X_2 are injective.
(c) Show that $X_1 \oplus X_2$ is flat if and only if X_1 and X_2 are flat.

Proof of (a). Let X_1 and X_2 be R-modules. Note that for for any R-module M,

$$\Phi_M : \mathrm{Hom}_R(X_1 \oplus X_2, M) \to \mathrm{Hom}_R(X_1, M) \oplus \mathrm{Hom}_R(X_2, M),$$

given by the isomorphism $\Phi_M(\pi) = (\pi_1, \pi_2)$ where $\pi_1(x_1) = \pi(x_1, 0)$ and $\pi_2(x_2) = \pi(0, x_2)$.

Moreover, for any R-module homomorphism $\psi : B \to C$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_R(X_1 \oplus X_2, B) & \xrightarrow{\quad\psi'\quad} & \mathrm{Hom}_R(X_1 \oplus X_2, C) \\ \downarrow{\scriptstyle \Phi_B} & & \downarrow{\scriptstyle \Phi_C} \\ \mathrm{Hom}_R(X_1, B) \oplus \mathrm{Hom}_R(X_2, B) & \xrightarrow{(\psi',\psi')} & \mathrm{Hom}_R(X_1, C) \oplus \mathrm{Hom}_R(X_2, C) \end{array}$$

To check, suppose $\pi \in \mathrm{Hom}_R(X_1 \oplus X_2, B)$ and $(x_1, x_2) \in X_1 \oplus X_2$. Let (π_1, π_2) denote $\Phi_B(\pi)$. Then evaluation $\Phi_C \circ \psi'(\pi)$ at $(x_1, x_2) \in X_1 \oplus X_2$ gives

$$\Phi_C(\psi(\pi(x_1, x_2)) = \Phi_C(\psi(\pi_1(x_1) + \pi_2(x_2))) = \Phi_C(\psi(\pi_1(x_1)) + \psi(\pi_2(x_2))) = (\psi \circ \pi_1, \psi \circ \pi_2)(x_1, x_2).$$

This is exactly the evaluation of $(\psi', \psi') \circ \Phi_B(\pi) = (\psi \circ \pi_1, \psi \circ \pi_2)$ at (x_1, x_2). Therefore the diagram commutes.

Now suppose that $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is a short exact sequence. Note that the map

$$\mathrm{Hom}_R(X_1, B) \oplus \mathrm{Hom}_R(X_2, B) \xrightarrow{(\psi',\psi')} \mathrm{Hom}_R(X_1, C) \oplus \mathrm{Hom}_R(X_2, C)$$

is surjective if any only if for each $i = 1, 2$ the map $\mathrm{Hom}_R(X_i, B) \xrightarrow{\psi'} \mathrm{Hom}_R(X_i, C)$ is surjective. By the isomorphism of sequences above, this shows that the induced map $\mathrm{Hom}_R(X_1 \oplus X_2, B) \xrightarrow{\psi'} \mathrm{Hom}_R(X_1 \oplus X_2, C)$ is surjective if and only if for both $i = 1, 2$, $\mathrm{Hom}_R(X_i, B) \xrightarrow{\psi'} \mathrm{Hom}_R(X_i, C)$ is surjective. Then by definition, $X_1 \oplus X_2$ is projective if and only if both X_1 and X_2 are projective. □

Proof of (b). Let X_1 and X_2 be R-modules, and let π_i be the projection map from $X_1 \oplus X_2$ onto X_i. Let $\varphi\colon A \to B$ be an injective homomorphism.

Suppose first that $X_1 \oplus X_2$ is injective, and choose any $\alpha_i\colon A \to X_i$ for $i = 1, 2$. Then define $\alpha \in \mathrm{Hom}_R(A, X_1 \oplus X_2)$ by $\alpha(a) = (\alpha_1(a), \alpha_2(a))$. By injectivity of $X_1 \oplus X_2$, we can lift α to a map $\beta \in \mathrm{Hom}_R(B, X_1 \oplus X_2)$ with $\alpha = \beta \circ \varphi$. Then $\alpha_i = \pi_i \circ \alpha = \pi_i \circ \beta \circ \varphi$, so $\pi_i \circ \beta \in \mathrm{Hom}_R(B, X_i)$ is the desired lift of α_i to B. It follows that X_1 is injective (and similarly so is X_2).

Conversely, suppose X_1 and X_2 are both injective, and let $\alpha\colon A \to X_1 \oplus X_2$. For $i = 1, 2$, write $\alpha_i = \pi_i \circ \alpha \in \mathrm{Hom}_R(A, X_i)$. By injectivity of X_i, we can lift α_i to a map $\beta_i\colon B \to X_i$, so that $\alpha_i = \beta_i \circ \varphi$. Then $\beta = (\beta_1, \beta_2) \in \mathrm{Hom}_R(B, X_1 \oplus X_2)$ satisfies $\alpha = \beta \circ \varphi$, so it is a lift of α. Thus $X_1 \oplus X_2$ is injective. □

Proof of (c). Let X_1 and X_2 be R-modules, and let π_i be the projection map from $X_1 \oplus X_2$ onto X_i. Let $\varphi\colon A \to B$ be any injection.

Recall that for any R-module M, there is an isomorphism $\Phi_M : (X_1 \oplus X_2) \otimes M \to (X_1 \otimes A) \oplus (X_2 \otimes A)$ with $\Phi_M((x_1, x_2) \otimes m)) = (x_1 \otimes m, x_2 \otimes m)$.

We claim that the following diagram commutes, where the horizontal arrows are induced from φ and the vertical arrows are the isomorphisms Φ_A and Φ_B:

$$\begin{array}{ccc}
(X_1 \oplus X_2) \otimes A & \xrightarrow{(1\oplus 1)\otimes\varphi} & (X_1 \oplus X_2) \otimes B \\
\big\downarrow \Phi_A & & \big\downarrow \Phi_B \\
(X_1 \otimes A) \oplus (X_2 \otimes A) & \xrightarrow{(1\otimes\varphi, 1\otimes\varphi)} & (X_1 \otimes B) \oplus (X_2 \otimes B)
\end{array}$$

By linearity, it suffices to check this on simple tensors. Let $(x_1, x_2) \otimes a \in (X_1 \oplus X_2) \otimes A$. Then

$$\Phi_B((1\oplus 1)\otimes\varphi)((x_1, x_2)\otimes a)) = \Phi_B((x_1, x_2)\otimes\varphi(a)) = (x_1\otimes\varphi(a), x_2\otimes\varphi(a)) = (1\otimes\varphi, 1\otimes\varphi)(\Phi_A(x_1, x_2)).$$

The top map $(X_1 \oplus X_2) \otimes A \xrightarrow{(1\oplus 1)\otimes\varphi} (X_1 \oplus X_2) \otimes B$ is injective if and only if the bottom map $(X_1 \otimes A) \oplus (X_2 \otimes A) \xrightarrow{(1\otimes\varphi, 1\otimes\varphi)} (X_1 \otimes B) \oplus (X_2 \otimes B)$. This map is injective if and only if for each $i = 1, 2$, $(X_i \otimes A) \xrightarrow{1\otimes\varphi} (X_i \otimes B)$ is injective. Therefore $X_1 \oplus X_2$ is flat if and only if both X_1 and X_2 are flat. □

Problem 2 (DF Exercise 14)**.** Let $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ be a sequence of R modules.

(a) Show that

$$0 \to \mathrm{Hom}_R(X, A) \xrightarrow{\varphi'} \mathrm{Hom}_R(X, B) \xrightarrow{\psi'} \mathrm{Hom}_R(X, C) \to 0$$

is a short exact sequence of abelian groups for all R-modules X if and only if the original sequence is a split short exact sequence. (Hint: consider $X = A$.)

(b) Show that

$$0 \to \mathrm{Hom}_R(C, X) \xrightarrow{\psi'} \mathrm{Hom}_R(B, X) \xrightarrow{\varphi'} \mathrm{Hom}_R(A, X) \to 0$$

is a short exact sequence of abelian groups for all R-modules X if and only if the original sequence is a split short exact sequence.

Proof of (a). Suppose that

$$0 \to \mathrm{Hom}_R(X, A) \xrightarrow{\varphi'} \mathrm{Hom}_R(X, B) \xrightarrow{\psi'} \mathrm{Hom}_R(X, C) \to 0$$

is exact for every R-module X and consider the module $X = R$. For every R-module M, $\Phi_M : \mathrm{Hom}_R(X, M) \to M$ given by $\Phi(f) = f(1)$ is an isomorphism. For any $f \in \mathrm{Hom}_R(R, A)$,

$$\Phi_B(\varphi'(f)) = \varphi'(f)(1) = \varphi(f(1)) = \varphi(\Phi_A(f)).$$

A similar argument for ψ shows that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(R,A) & \xrightarrow{\varphi'} & \mathrm{Hom}_R(R,B) & \xrightarrow{\psi'} & \mathrm{Hom}_R(R,C) & \longrightarrow & 0 \\
 & & \downarrow \Phi_A & & \downarrow \Phi_B & & \downarrow \Phi_C & & \\
0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0
\end{array}$$

Since Φ_A, Φ_B, Φ_C are isomorphisms and the top row is an exact sequence, the bottom row is as well.

Next, consider the module $X = C$ and consider the homomorphism $\mathrm{id}_C \in \mathrm{Hom}_R(C,C)$. Because ψ' is surjective, there exists $\mu \in \mathrm{Hom}_R(C,B)$ for which $\mathrm{id}_C = \psi'(\mu) = \psi \circ \mu$. This shows that the exact sequence $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ splits.

Conversely, suppose that the sequence $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ splits, meaning that $\mu \in \mathrm{Hom}_R(C,B)$ for which $\mathrm{id}_C = \psi \circ \mu$. Suppose that X is an R-module and $g \in \mathrm{Hom}_R(X,C)$. Then $f = \mu \circ g$ belongs to $\mathrm{Hom}_R(X,B)$ and $\psi \circ f = g$. Therefore the map $\psi' : \mathrm{Hom}_R(X,B) \to \mathrm{Hom}_R(X,C)$ is surjective, making the sequence

$$0 \to \mathrm{Hom}_R(X,A) \xrightarrow{\varphi'} \mathrm{Hom}_R(X,B) \xrightarrow{\psi'} \mathrm{Hom}_R(X,C) \to 0$$

is exact. □

Proof of (b). Suppose that

$$0 \to \mathrm{Hom}_R(C,X) \xrightarrow{\psi'} \mathrm{Hom}_R(B,X) \xrightarrow{\varphi'} \mathrm{Hom}_R(A,X) \to 0$$

is exact for every R-module X.

(ψ surjective) To see ψ is surjective: let $X = C/\psi(B)$ and let $\pi\colon C \to C/\psi(B)$ be the corresponding quotient map $\pi(c) = c + \psi(B)$. By assumption, we have that $\mathrm{Hom}_R(C,X) \xrightarrow{\psi'} \mathrm{Hom}_R(B,X)$ is injective. Since $\pi(\psi(b)) = 0$ for all $b \in B$, we see that $\psi'(\pi) = \pi \circ \psi = 0$. By injectivity of ψ', we must have $\pi = 0$ and hence $C/\psi(B) = \{0\}$, giving $C = \psi(B)$.

($\mathrm{im}(\varphi) \subseteq \ker(\psi)$) By assumption, $\varphi' \circ \psi' = 0$, that is, $f \circ \psi \circ \varphi = 0 \in \mathrm{Hom}_R(A,X)$ for all $f \in \mathrm{Hom}_R(C,X)$. Taking $X = C$ and $f = \mathrm{id}_C$ then gives $\psi \circ \varphi = 0$.

($\ker(\psi) \subseteq \mathrm{im}(\varphi)$) Let $X = B/\varphi(A)$ and let $\pi \in \mathrm{Hom}_R(B,X)$ be the corresponding quotient map $\pi(b) = b + \varphi(A)$. Then $\pi \circ \varphi = \varphi'(\pi) = 0$, so by assumption, there must exist some element $\sigma \in \mathrm{Hom}_R(C,X)$ such that $\psi'(\sigma) = \sigma \circ \psi = \pi$. Then $\mathrm{im}(\varphi) = \ker(\pi)$ must contain the kernel of ψ.

(φ injective) Let $X = A$. The map $\varphi'\colon \mathrm{Hom}_R(B,A) \to \mathrm{Hom}_R(A,A)$ is surjective. In particular, id_A lies in the image of φ', so there exists $\mu \in \mathrm{Hom}_R(B,A)$ such that $\mathrm{id}_A = \varphi'(\mu) = \mu \circ \varphi$. It follows that φ must be injective, and by Proposition 10.5.26, the exact sequence $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ splits.

Conversely, suppose $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is a split exact sequence. By Theorem 10.5.28, the sequence $0 \to \mathrm{Hom}_R(C,X) \xrightarrow{\psi'} \mathrm{Hom}_R(B,X) \xrightarrow{\varphi'} \mathrm{Hom}_R(A,X)$ is exact. It suffices to show that φ' is surjective. Since the sequence above splits, by Proposition 10.5.26, there exists $\lambda \in \mathrm{Hom}_R(B,A)$ so that $\mathrm{id}_A = \lambda \circ \varphi$. For any $g \in \mathrm{Hom}_R(A,X)$, let $f = g \circ \lambda \in \mathrm{Hom}_R(B,X)$.

Since $\varphi'(f) = f \circ \varphi = g \circ \lambda \circ \varphi = g$, it follows that φ' is surjective, so

$$0 \to \mathrm{Hom}_R(C, X) \xrightarrow{\psi'} \mathrm{Hom}_R(B, X) \xrightarrow{\varphi'} \mathrm{Hom}_R(A, X) \to 0$$

must be exact. $\square$