Problem 1 (DF Exercise 16 +). Suppose that \( R \) is a commutative ring with \( 1 \neq 0 \) and let \( I \) and \( J \) be ideals of \( R \), so \( R/I \) and \( R/J \) are naturally \( R \)-modules.

(a) Prove that every element of \( R/I \otimes_R R/J \) can be written as a simple tensor 
\[
(1 \mod I) \otimes (r \mod J).
\]

(b) Prove that there is an \( R \)-module isomorphism \( R/I \otimes_R R/J \to R/(I + J) \) mapping 
\[
(r \mod I) \otimes (s \mod J) \to (rs \mod I + J).
\]

(c) Give an example of \( R, I, J \) and an element in \( (R/I)^2 \otimes_R (R/J)^2 \) that cannot be written as a simple tensor. Make sure to justify your answer.

Definition. Let \( R \) be a commutative ring with identity \( 1_R \neq 0 \). An \( R \)-algebra is a ring \( A \) with identity \( 1_A \neq 0 \) and a ring homomorphism \( f: R \to A \) with

1. \( f(1_R) = 1_A \) and
2. \( f(r)a = af(r) \) for all \( r \in R \) and \( a \in A \).

In particular, if \( R \) is a subring of \( A \) contained in its center with \( 1_R = 1_A \), then \( A \) is an \( R \)-algebra with the map \( f: R \to A \) given by \( f(r) = r \).

Problem 2. Let \( R, A, B \) be rings with \( R \) contained in the center of \( A \) and the center of \( B \) and with coinciding (nonzero) multiplicative identities \( 1_R = 1_A = 1_B \neq 0 \).

(a) Show that the multiplication \( (a \otimes b)(a' \otimes b') = aa' \otimes bb' \) makes \( A \otimes_R B \) into an \( R \)-algebra. (In the proof of Proposition 10.4.21, it is shown that this multiplication is well-defined. This completes the proof of the statement of this proposition.)

(b) Show that \( \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C} \) as rings.