Math 721 – Homework 2

Due Thursday, January 23, 2019 at 10:15am

Good practice problems (do not turn in solutions):
DF 10.4 Exercises 3, 4, 5, 10, 11, 17, 21

Problem 1 (DF Exercise 16 +)**.** Suppose that R is a commutative ring with $1 \neq 0$ and let I and J be ideals of R, so R/I and R/J are naturally R-modules.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor $(1 \mod I) \otimes (r \mod J)$.

(b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \to R/(I+J)$ mapping $(r \mod I) \otimes (s \mod J)$ to $(rs \mod I+J)$.

(c) Give an example of R, I, J and an element in $(R/I)^2 \otimes_R (R/J)^2$ that cannot be written as a simple tensor. Make sure to justify your answer.

Definition. Let R be a commutative ring with identity $1_R \neq 0$. An R-*algebra* is a ring A with identity $1_A \neq 0$ and a ring homomorphism $f : R \to A$ with

(1) $f(1_R) = 1_A$ and

(2) $f(r)a = af(r)$ for all $r \in R$ and $a \in A$.

In particular, if R is a subring of A contained in its center with $1_R = 1_A$, then A is an R-algebra with the map $f : R \to A$ given by $f(r) = r$.

Problem 2. Let R, A, B be rings with R contained in the center of A and the center of B and with coinciding (nonzero) multiplicative identities $1_R = 1_A = 1_B \neq 0$.

(a) Show that the multiplication $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ makes $A \otimes_R B$ into an R-algebra. (In the proof of Proposition 10.4.21, it is shown that this multiplication is well-defined. This completes the proof of the statement of this proposition.)

(b) Show that $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.