Problem 1 (DF 18.1.15 & 16). Let $G$ be a finite abelian group. This exercise concerns 1-dimensional complex representations of $G$. (Here “complex” means “over $\mathbb{C}$”.)

(a) If $G$ is cyclic, exhibit all 1-dimensional complex representations of $G$. Make sure to decide which are inequivalent.

(b) For arbitrary finite abelian group $G$, exhibit all 1-dimensional complex representations of $G$ and decide which are inequivalent.

(c) Conclude that the number of inequivalent 1-dimensional complex representations of $G$ equals $|G|$.

Proof of (a). Let $G = \langle \sigma : \sigma^n = 1 \rangle$ be the cyclic group of order $n$. Note that a group homomorphism $\varphi : G \to \text{GL}_1(\mathbb{C}) = \mathbb{C}^*$ is determined by $\varphi(\sigma)$, since $\varphi(\sigma^k) = \varphi(\sigma)^k$ for all $k \in \mathbb{Z}$. Note that

$$\varphi(\sigma)^n = \varphi(\sigma^n) = \varphi(1) = 1,$$

so $\varphi(\sigma)$ must be an $n$th root of unity. Let $\omega$ be a primitive $n$th roots of unity (e.g. $\omega = e^{2\pi i/n}$). Then $\varphi(\sigma) = \omega^{\ell}$ for some $\ell \in \{0, 1, \ldots, n-1\}$. Moreover, for any $\ell \in \{0, 1, \ldots, n-1\}$, $\psi(\sigma^k) = \omega^{\ell k}$ defines a group homomorphism $\psi : G \to \mathbb{C}^*$.

Now consider a representation $\varphi : G \to \text{GL}_1(\mathbb{C}) = \mathbb{C}^*$. All equivalent representations $\psi$ have the form

$$\psi(g) = u^{-1} \varphi(g) u \quad \text{for some } u \in \text{GL}_1(\mathbb{C}) = \mathbb{C}^*.$$

Since the multiplicative group of $\mathbb{C}^*$ is abelian, we see that

$$\psi(g) = u^{-1} \varphi(g) u = \varphi(g) u^{-1} = \varphi(g).$$

Therefore the only way for two 1-dimensional representations to be equivalent is if they are the same. \hfill \Box

Proof of (b). Let $G$ be a finite abelian group. By the classification of finite abelian groups (i.e. torsion $\mathbb{Z}$-modules), we can write $G = G_1 \oplus \ldots \oplus G_r$ where each component $G_j \cong \mathbb{Z}/n_j \mathbb{Z}$ is cyclic of size $n_j \in \mathbb{Z}_+$. Then the set of group homomorphisms from $G$ to $\mathbb{C}^*$ satisfies

$$\text{Hom}_\mathbb{Z}(G, \mathbb{C}^*) = \text{Hom}_\mathbb{Z}(G_1 \oplus \ldots \oplus G_r, \mathbb{C}^*) \cong \text{Hom}_\mathbb{Z}(G_1, \mathbb{C}^*) \oplus \ldots \oplus \text{Hom}_\mathbb{Z}(G_r, \mathbb{C}^*).$$

Explicitly, for each $j = 1, \ldots, r$, let $\sigma_j$ denote the element of $G$ corresponding given the identity in the $G_j$th factor and zero in the rest, so that $G_j = \langle \sigma_j \rangle$ is a cyclic group of order $n_j$. Any group homomorphism $\varphi : G \to \mathbb{C}^*$ is uniquely determined by its values $\sigma_1, \ldots, \sigma_r$. Since $\varphi$ restricts to a homomorphism $\varphi|_{G_j} : G_j \to \mathbb{C}^*$, the options for $\varphi(\sigma_j)$ are exactly the $n_j$th roots of unity. If $\omega_j = e^{2\pi i/n_j}$ is a primitive $n_j$th root of unity in $\mathbb{C}$, then for every choice of $(\ell_1, \ldots, \ell_r) \in \mathbb{Z}_{\geq 0}^r$ with $0 \leq \ell_j \leq n_j - 1$, there is a unique 1-dimensional complex representation $\varphi(\sigma_j) = (\omega_j)^{\ell_j}$.

As argued in part (a), the only way for two 1-dimensional representations to be equivalent is if they are the same. \hfill \Box

Proof of (c). In part (b), we saw that if $G$ is a finite abelian group then it is isomorphism to $\mathbb{Z}/n_1 \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_r \mathbb{Z}$ for some $n_1, \ldots, n_r \in \mathbb{Z}_+$, giving $|G| = \prod_{j=1}^r n_j$ and the number of inequivalent 1-dimensional complex representations of $G$ is $|\text{Hom}_\mathbb{Z}(G, \mathbb{C}^*)| = \prod_{j=1}^r n_j$. \hfill \Box
Problem 2 (DF 18.2.12). Let \( F \) be a field, \( f(x) \in F[x] \), and \( R = F[x]/f(x) \).

(a) Find necessary and sufficient conditions on the factorization of \( f(x) \) in \( F[x] \) so that \( R \) is a semisimple ring.

(b) When \( R \) is semisimple, describe its Wedderburn decomposition.

Proof of (a). Let \( f = f_1^{\alpha_1} \cdots f_r^{\alpha_r} \) be a factorization of \( f \) in \( F[x] \), where \( f_1, \ldots, f_r \) are irreducible and distinct and \( \alpha_1, \ldots, \alpha_r \in \mathbb{Z}_+ \). Let \( R = F[x]/\langle f(x) \rangle \). By the Chinese remainder theorem,
\[
R \cong F[x]/\langle f_1^{\alpha_1} \rangle \times \cdots \times F[x]/\langle f_r^{\alpha_r} \rangle.
\]

Claim 1. The ring \( R = F[x]/f(x) \) is semisimple if and only if \( \alpha_j = 1 \) for all \( j = 1, \ldots, r \).

\((\Leftarrow)\) If \( \alpha_j = 1 \) for all \( j = 1, \ldots, r \), then
\[
R \cong F[x]/\langle f_1 \rangle \times \cdots \times F[x]/\langle f_r \rangle.
\]

For every \( j \), \( f_j \) is irreducible, meaning that \( F_j = F[x]/\langle f_j \rangle \) is a field. Any field is isomorphic to the ring of \( 1 \times 1 \) matrices over that field, so we find that
\[
R \cong M_1(F_1) \times \cdots \times M_1(F_r).
\]

By criterion # 5 in Wedderburn’s theorem, \( R \) is semisimple.

\((\Rightarrow)\) Suppose that \( R \) is semisimple. Its Wedderburn decomposition is its Wedderburn decomposition has the form
\[
R \cong M_{n_1}(\Delta_1) \times \cdots \times M_{n_r}(\Delta_r).
\]

for some \( n_1, \ldots, n_r \in \mathbb{Z}_+ \) and division rings \( \Delta_1, \ldots, \Delta_r \). Since \( R \) is commutative, we see that \( n_j = 1 \) for all \( j \) and \( \Delta_j \) must be commutative and hence a field \( F_j \). This gives the ring isomorphisms
\[
R \cong M_1(F_1) \times \cdots \times M_1(F_r) \cong F_1 \times \cdots \times F_r.
\]

It follows that \( R \) has no nonzero nilpotent elements. To see this, suppose that \( a = (a_1, \ldots, a_r) \) is an element of \( F_1 \times \cdots \times F_r \), with \( 0 = a^n = (a_1^n, \ldots, a_r^n) \) for some \( n \in \mathbb{Z}_+ \). Then \( a_j^n = 0 \) for all \( n \). Since each \( F_j \) is a field, \( a_j^n = 0 \) implies that \( a_j = 0 \). It follows that \( a = (0, \ldots, 0) = 0 \).

Suppose, for the sake of contradiction, that in the irreducible factorization of \( f \), \( \alpha_j > 1 \) for some \( j \). Let \( e_j \) denote the element of \( R \) that equals 1 modulo \( \langle f_j^{\alpha_j} \rangle \) and 0 modulo \( \langle f_i^{\alpha_i} \rangle \) for all \( i \neq j \). Since \( \alpha_j > 1 \), \( f_j e_j \) is nonzero in \( R \), but we see that \( (f_j e_j)^{\alpha_j} = f_j^{\alpha_j} e_j = 0 \). Therefore \( f_j e_j \) is a nonzero nilpotent element in \( R \) and giving a contradiction.

This shows that if \( R \) is semisimple, then \( \alpha_j = 1 \) for all \( j = 1, \ldots, r \) in the irreducible decomposition of \( f \). \( \square \)

Solution to (b). If \( R \) is semisimple, then the Wedderburn decomposition of \( R \) is given in Equation [1] in part (a) above. \( \square \)