Math 721 – Homework 10 – Solutions

Problem 1 (DF 18.1.15 & 16)**.** Let G be a finite abelian group. This exercise concerns 1-dimensional complex representations of G. (Here "complex" means "over $\mathbb{C}$".)

(a) If G is cyclic, exhibit all 1-dimensional complex representations of G. Make sure to decide which are inequivalent.

(b) For arbitrary finite abelian group G, exhibit all 1-dimensional complex representations of G and decide which are inequivalent.

(c) Conclude that the number of inequivalent 1-dimensional complex representations of G equals $|G|$.

Proof of (a). Let $G = \langle \sigma : \sigma^n = 1 \rangle$ be the cyclic group of order n. Note that a group homomorphism $\varphi : G \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$ is determined by $\varphi(\sigma)$, since $\varphi(\sigma^k) = \varphi(\sigma)^k$ for all $k \in \mathbb{Z}$. Note that

$$\varphi(\sigma)^n = \varphi(\sigma^n) = \varphi(1) = 1,$$

so $\varphi(\sigma)$ must be an nth root of unity. Let ω be a primitive nth roots of unity (e.g $\omega = e^{2\pi i/n}$). Then $\varphi(\sigma) = \omega^\ell$ for some $\ell \in \{0, 1, \dots n-1\}$. Moreover, for any $\ell \in \{0, 1, \dots n-1\}$, $\varphi(\sigma^k) = \omega^k \ell$ defines a group homomorphism $\varphi : G \to \mathbb{C}^*$.

Now consider a representation $\varphi : G \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$. All equivalent representations ψ have the form

$$\psi(g) = u^{-1}\varphi(g)u \quad \text{for some } u \in \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*.$$

Since the multiplicative group of $\mathbb{C}^*$ is abelian, we see that

$$\psi(g) = u^{-1}\varphi(g)u = \varphi(g)u^{-1}u = \varphi(g).$$

Therefore the only way for two 1-dimensional representations to be equivalent is if they are the same. $\square$

Proof of (b). Let G be a finite abelian group. By the classification of finite abelian groups (i.e. torsion $\mathbb{Z}$-modules), we can write $G = G_1 \oplus \dots \oplus G_r$ where each component $G_j \cong \mathbb{Z}/n_j\mathbb{Z}$ is cyclic of size $n_j \in \mathbb{Z}_+$. Then the set of group homomorphisms from G to $\mathbb{C}^*$ satisfies

$$\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^*) = \mathrm{Hom}_{\mathbb{Z}}(G_1 \oplus \dots \oplus G_r, \mathbb{C}^*) \cong \mathrm{Hom}_{\mathbb{Z}}(G_1, \mathbb{C}^*) \oplus \dots \oplus \mathrm{Hom}_{\mathbb{Z}}(G_r, \mathbb{C}^*).$$

Explicitly, for each $j = 1, \dots, r$, let σ_j denote the element of G corresponding given the identity in the G_jth factor and zero in the rest, so that $G_j = \langle \sigma_j \rangle$ is a cyclic group of order n_j. Any group homomorphism $\varphi : G \to \mathbb{C}^*$ is uniquely determined by its values $\sigma_1, \dots, \sigma_r$. Since φ restricts to a homomorphism $\varphi|_{G_j} : G_j \to \mathbb{C}^*$, the options for $\varphi(\sigma_j)$ are exactly the n_jth roots of unity. If $\omega_j = e^{2\pi i/n_j}$ is a primitive n_jth root of unity in $\mathbb{C}$, then for every choice of $(\ell_1, \dots, \ell_r) \in \mathbb{Z}^r_{\geq 0}$ with $0 \leq \ell_j \leq n_j - 1$, there is a unique 1-dimensional complex representation $\varphi(\sigma_j) = (\omega_j)^{\ell_j}$.

As argued in part (a), the only way for two 1-dimensional representations to be equivalent is if they are the same. $\square$

Proof of (c). In part (b), we saw that if G is a finite abelian group then it is isomorphism to $\mathbb{Z}/n_1\mathbb{Z} \oplus \dots \mathbb{Z}/n_r\mathbb{Z}$ for some $n_1, \dots, n_r \in \mathbb{Z}_+$, giving $|G| = \prod_{j=1}^r n_j$ and the number of inequivalent 1-dimensional complex representations of G is $|\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^*)| = \prod_{j=1}^r n_j$. $\square$

Problem 2 (DF 18.2.12)**.** Let F be a field, $f(x) \in F[x]$, and $R = F[x]/f(x)$.

(a) Find necessary and sufficient conditions on the factorization of $f(x)$ in $F[x]$ so that R is a semisimple ring.

(b) When R is semisimple, describe its Wedderburn decomposition.

Proof of (a). Let $f = f_1^{\alpha_1} \cdots f_r^{\alpha_r}$ be a factorization of f in $F[x]$, where $f_1, \ldots, f_r$ are irreducible and distinct and $\alpha_1, \ldots, a_r \in \mathbb{Z}_+$. Let $R = F[x]/\langle f(x)\rangle$. By the Chinese remainder theorem,

$$R \cong F[x]/\langle f_1^{\alpha_1}\rangle \times \ldots \times F[x]/\langle f_r^{\alpha_r}\rangle.$$

Claim 1. *The ring $R = F[x]/f(x)$ is semisimple if and only if $\alpha_j = 1$ for all $j = 1, \ldots, r$.*

($\Leftarrow$) If $\alpha_j = 1$ for all $j = 1, \ldots, r$, then

$$R \cong F[x]/\langle f_1\rangle \times \ldots \times F[x]/\langle f_r\rangle.$$

For every j, f_j is irreducible, meaning that $F_j = F[x]/\langle f_j\rangle$ is a field. Any field is isomorphic to the ring of 1×1 matrices over that field, so we find that

$$R \cong M_1(F_1) \times \ldots \times M_1(F_r). \tag{1}$$

By criterion # 5 in Wedderburn's theorem, R is semisimple.

($\Rightarrow$) Suppose that R is semisimple. Its Wedderburn decomposition is its Wedderburn decomposition has the form

$$R \cong M_{n_1}(\Delta_1) \times \ldots \times M_{n_r}(\Delta_r).$$

for some $n_1, \ldots, n_r \in \mathbb{Z}_+$ and division rings $\Delta_1, \ldots, \Delta_r$. Since R is commutative, we see that $n_j = 1$ for all j and Δ_j must be commutative and hence a field F_j. This gives the ring isomorphisms

$$R \;\cong\; M_1(F_1) \times \ldots \times M_1(F_r) \;\cong\; F_1 \times \ldots \times F_r.$$

It follows that R has no nonzero nilpotent elements. To see this, suppose that $a = (a_1, \ldots, a_r)$ is al element of $F_1 \times \ldots \times F_r$ with $0 = a^n = (a_1^n, \ldots, a_r^n)$ for some $n \in \mathbb{Z}_+$. Then $a_j^n = 0$ for all n. Since each F_j is a field, $a_j^n = 0$ implies that $a_j = 0$. It follows that $a = (0, \ldots, 0) = 0$.

Suppose, for the sake of contradiction, that in the irreducible factorization of f, $\alpha_j > 1$ for some j. Let e_j denote the element of R that equals 1 modulo $\langle f_j^{\alpha_j}\rangle$ and 0 modulo $\langle f_i^{\alpha_i}\rangle$ for all $i \neq j$. Since $\alpha_j > 1$, $f_j e_j$ is nonzero in R, but we see that $(f_j e_j)^{\alpha_j} = f_j^{\alpha_j} e_j = 0$. Therefore $f_j e_j$ is a nonzero nilpotent element in R and giving a contradiction.

This shows that if R is semisimple, then $\alpha_j = 1$ for all $j = 1, \ldots, r$ in the irreducible decomposition of f. $\square$

Solution to (b). If R is semisimple, then the Wedderburn decomposition of R is given in Equation (1) in part (a) above. $\square$