Math 721 – Homework 1

Due Thursday, January 16, 2019 at 10:15am

Good practice problems (do not turn in solutions):
DF 10.1 Exercises 2, 4, 7, 9, 10, 11
DF 10.2 Exercises 3, 5, 6, 12
DF 10.3 Exercises 2, 7, 9, 16, 17

Problem 1 (Kernels and images)**.** Let R be a ring with $1 \neq 0$.

(a) If $\varphi : M \to N$ is a homomorphism of R-modules, show that the kernel and image of φ are submodules of M and N, respectively.

(b) For each of the following R-module homomorphisms, describe the kernel and image as simply as possible. (You do not need to check that they are homomorphisms.)

(i) $R = \mathbb{Q}[x, y]$, $M = R^2$, $N = R$, $\varphi(f, g) = xf + yg$,

(ii) $R = \mathbb{Z}$, $M = R^2$, $N = R$, $\varphi(a, b) = 4a + 6b$, and

(ii) $R = \mathrm{Mat}_{2\times 2}(\mathbb{Q})$, $M = \mathrm{Mat}_{2\times 3}(\mathbb{Q})$, $N = \mathrm{Mat}_{2\times 4}(\mathbb{Q})$, $\varphi(A) = AU$ where

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Problem 2 (Hom_R)**.** Let R be a commutative ring with $1 \neq 0$ and let A, B and M be R-modules. Show the following isomorphisms of R-modules:

(a) $\mathrm{Hom}_R(R, M) \cong M$,

(b) $\mathrm{Hom}_R(A \times B, M) \cong \mathrm{Hom}_R(A, M) \times \mathrm{Hom}_R(B, M)$, and

(c) $\mathrm{Hom}_R(R^n, M) \cong \underbrace{M \times \cdots \times M}_{n}$.

(d) Give an example to show that it is not always that case that $\mathrm{Hom}_R(M, R) \cong M$.